7102 Analysis 4: Real Analysis Notes

Based on the 2012 spring lectures by Dr N Sidorova

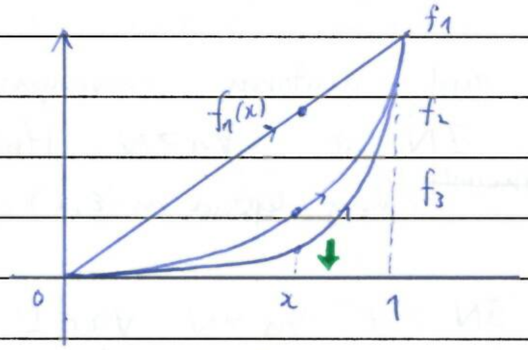
9/11

// < Uniform convergence >

Def: Let $I \subset \mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$, f are real-valued functions. We say that f_n converges to f pointwise if $\forall x \in I$, $f_n(x) \underset{n\to\infty}{\to} f(x)$

Example: $I = (0,1)$

$f_n(x) = x^n$



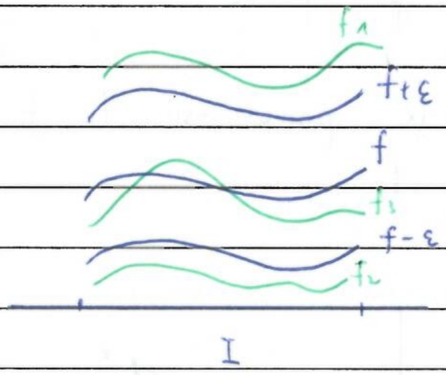
$\forall x \in (0,1)$, $f_n(x) = x^n \to 0$

So f_n converges pointwise to the zero function

// **Def**: Let $I \subset \mathbb{R}$ and let $\{f_n\}_{n=1}^{\infty}$, f be real-valued functions on I, we say that f_n converges to f uniformly if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$

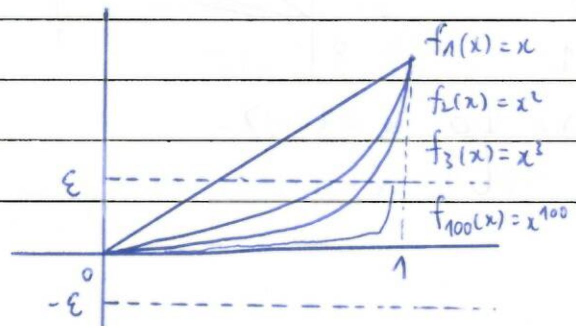
$\forall n \geq N$ $\forall x \in I$

$|f_n(x) - f(x)| < \varepsilon$



* For any ε- tube around f all functions f_n will eventually fit into ε- tube.

Example: $I = (0,1)$, $f_n(x) = x^n$



* f_n doesn't converge uniformly to 0

Thm 1.1: If $f_n \to f$ uniformly, then $f_n \to f$ pointwise

Proof: We want to prove $\forall x \in I$ $\underbrace{f_n(x) \to f(x)}$

$\Updownarrow$

(this follows from the definition of the uniform convergence) $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ st. $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$

□

Difference between pointwise and uniform convergence

Pointwise: $\forall x \in I$, $\forall \varepsilon > 0$ $\exists N$ st. $\forall n > N$ $|f_n(x) - f(x)| < \varepsilon$

~~(mainly~~ is (may depend on ε, x)

Uniform: $\forall \varepsilon > 0$ $\exists N$ st. $\forall n > N$, $\forall x \in I$ $|f_n(x) - f(x)| < \varepsilon$

depend only on ε

In the examples below, do the sequences converge pointwise / uniformly?

Plan ① : × study pointwise convergence

② : × if the sequence doesn't converge pointwise it doesn't converge uniformly.

③ × if the sequence does converge pointwise to f ~~it does~~ test converge uniformly to f

Study uniform convergence.

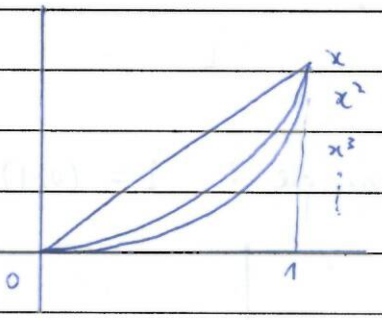
① $I = [0,1]$, $f_n(x) = x^n$

- pointwise ?

for $x \in (0,1)$, $f_n(x) = x^n \to 0$

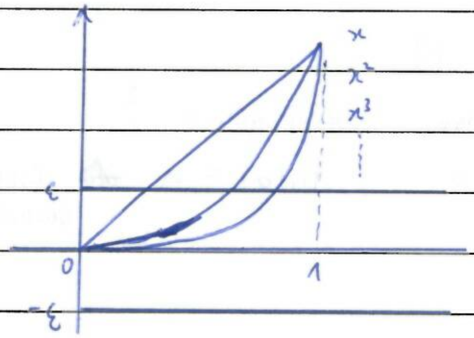
$x = 0$, $f_n(0) = 0 \to 0$

$x = 1$, $f_n(1) = 1 \to 1$



$\Rightarrow$ f_n converges pointwise to $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

- uniformly ? NO !



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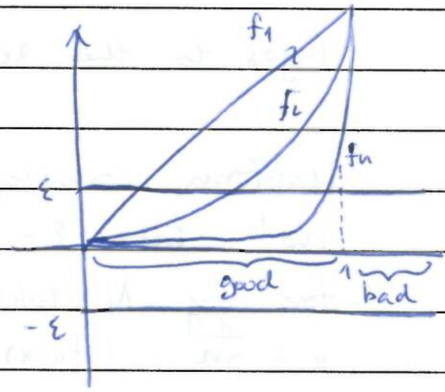
// Negation of uniform convergence

f_n doesn't converge to f uniformly on I if

$$\boxed{\underbrace{\exists\, \varepsilon > 0}_{\text{any small } \varepsilon}\quad \forall N \in \mathbb{N} \quad \underbrace{\exists\, n \geq N}_{\text{usually } n=N \text{ works}} \quad \exists\, x \in I \quad |f_n(x) - f(x)| \geq \varepsilon}$$

st. f_n don't stay in tube



Take $\varepsilon = \frac{1}{2}$, for any N, take $n = N$

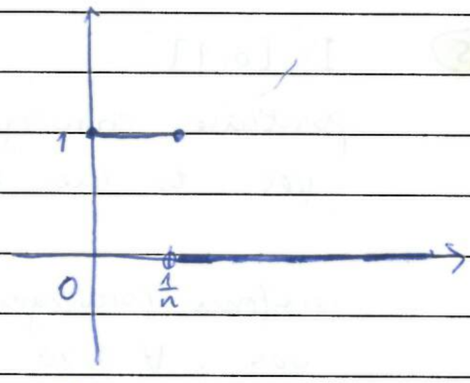
and x such that $x^N \geq \frac{1}{2}$

(ie. $x \in \left[\frac{1}{2^{\frac{1}{N}}} ; 1\right)$)

$$|f_n(x) - f(x)| = |x^N - 0| > \frac{1}{2} = \varepsilon$$

(2) $I = [0, \infty)$

$$f_n(x) = \begin{cases} 0 & x > \frac{1}{n} \\ 1 & 0 \leq x \leq \frac{1}{n} \end{cases}$$

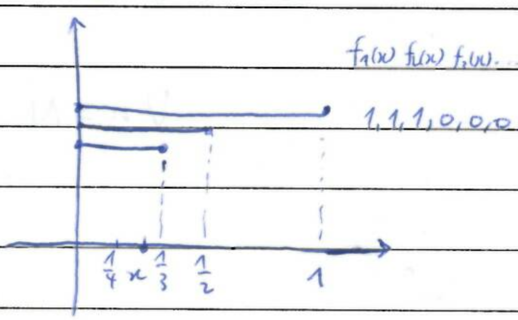


pointwise convergence ?

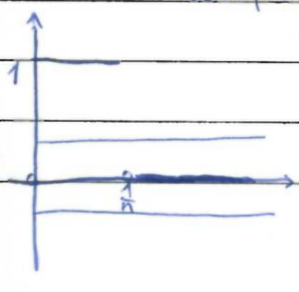
$x = 0$ $\quad f_n(0) = 1 \to 1$

$x > 0$ $\quad f_n(x) \to 0$

yes, the limit function is $f(x) = \begin{cases} 0 & , x > 0 \\ 1 & , x = 1 \end{cases}$



uniform convergence ?



Take $\varepsilon = \frac{1}{2}$, for any N

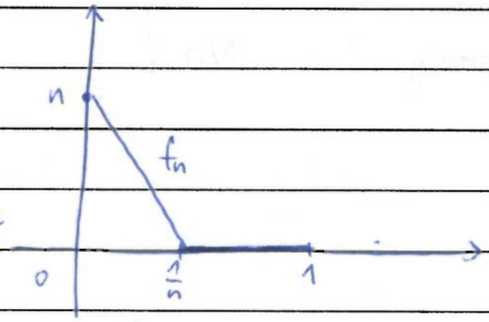
take $N = n$ and $x = \frac{1}{2n}$

$$|f_n(x) - f(x)| = |1 - 0| = 1 \geq \frac{1}{2} = \varepsilon$$

(3) $I = [0,1]$

pointwise convergence ?

$x = 0$, $f_n(0) = n$ ~~the~~ doesn't converge



no !

no uniform convergence either !
[by thm 1.1]

(4) $I = [0,1]$

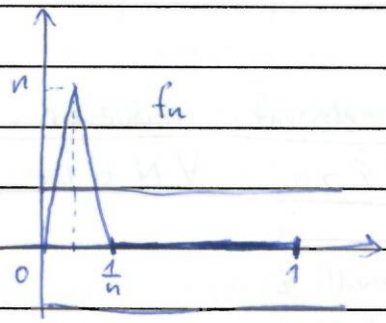
pointwise convergence ?

$x = 0$ $f_n(0) = 0 \to 0$

$x > 0$ $f_n(x) \to 0$

$* * * * \cdots 0000$

(yes to the zero function)



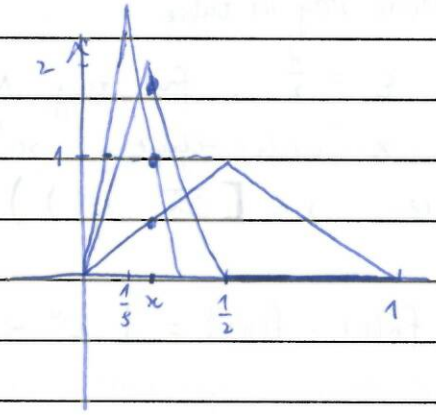
uniform convergence ?

no ! take $\varepsilon = \frac{1}{3}$,

for any N take $n = N$ and

$x = \frac{1}{2n}$: $|f_n(x) - f(x)| = |n - 0| =$

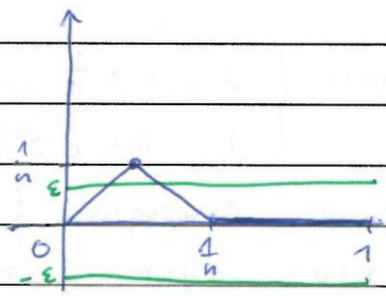
$n \geq \frac{1}{3} = \varepsilon$



(5) $I = [0,1]$

pointwise convergence :

yes , to the zero function



uniform convergence :

yes , $\forall \varepsilon > 0$, take N, so that $\frac{1}{N} < \varepsilon$ (for example $N = [\frac{1}{\varepsilon}] + 1$)

$$\forall n \geq N \quad \forall x \quad |f_n(x) - f(x)| = |\tfrac{1}{n} - 0| = \tfrac{1}{n} \leq \tfrac{1}{N} < \varepsilon$$

⑥ $I = [0, \infty)$; $f_n(x) = \frac{1}{n+x}$

pointwise convergence ?

$\forall x : \frac{1}{n+x} \longrightarrow 0$ as $n \to \infty$

yes, to the zero function

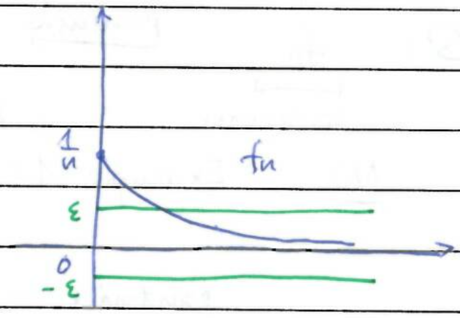
uniform convergence :

yes,

$\forall \varepsilon > 0$, choose N st. $\frac{1}{N} < \varepsilon$;

then $\forall n \geq N$ and $\forall x$

$|f_n(x) - f(x)| = \left|\frac{1}{n+x} - 0\right| = \frac{1}{n+x} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$



⑦ $I = (0, \infty)$; $f_n(x) = \frac{nx}{1+n+x}$

pointwise ? $f_n(x) = \frac{nx}{1+n+x} \xrightarrow{n \to \infty} x$

yes , to $f(x) = x$

uniform ? no !

// $f: [a,b] \to \mathbb{R}$

f is continuous at $x \in [a,b]$

if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\forall y \in [a,b]$ with $|y-x| < \delta$

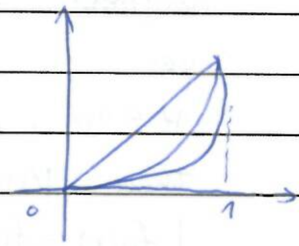
$|f(y) - f(x)| < \varepsilon$

Q: $\underbrace{f_n}_{\text{continuous}} \xrightarrow{\text{pointwise}} f$

$\uparrow$ continuous ?

NO Example 1: $[0,1]$

$f_n(x) = x^n$



pointwise limit is discontinuous!

// Thm 1.2: Let $\{f_n\}_{n=1}^{\infty}$, f be real-valued functions on $[a,b]$ and suppose that all f_n are continuous and $f_n \to f$ uniformly on $[a,b]$. Then f is continuous.

Proof: Let $x \in [a,b]$

Let $\varepsilon > 0$

By the uniform convergence

$\exists N$ st. $\forall n \geq N$, $\forall z \in [a,b]$, $|f_n(z) - f(z)| < \frac{\varepsilon}{3}$

Since f_N is continuous at x

$\exists \delta > 0$ st. $\forall y \in [a,b]$ with $|y-x| < \delta$ we have

$|f_N(y) - f_N(x)| < \frac{\varepsilon}{3}$

$$\Rightarrow |f(y) - f(x)| = |f(y) - f_N(y) + f_N(y) - f_N(x) + f_N(x) - f(x)|$$

$$\leq \underbrace{|f(y) - f_N(y)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(y) - f_N(x)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_N(x) - f(x)|}_{< \frac{\varepsilon}{3}} < \varepsilon$$

$\square$

Remark: If a sequence of continuous functions converges pointwise to a discontinuous function then the convergence is not uniform

// Compact sets in $\mathbb{R}$

(a, b) open interval, denoted I
$[a, b]$ closed intervals

Let $\{I_\alpha\}_{\alpha \in A}$ be a collection of open intervals

Example: (1) $I_1, I_2, \dots, I_m$

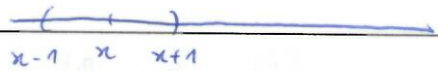
$\{I_i\}_{i \in \{1, \dots, m\}}$ $\qquad A = \{1, \dots, m\}$

(2) $I_n = (-n, n)$

$\{I_n\}_{n \in \mathbb{N}}$ $\qquad A = \mathbb{N}$

(3) $I_x = (x-1, x+1)$

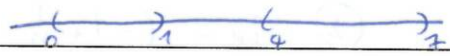
$\{I_x\}_{x \in \mathbb{R}}$ $\qquad A = \mathbb{R}$



Def: Let $S \subset \mathbb{R}$ be a set and $\{I_\alpha\}_{\alpha \in A}$ be a collection of open intervals.
We say that $\{I_\alpha\}_{\alpha \in A}$ is a cover of S if $S \subset \bigcup_{\alpha \in A} I_\alpha$

Examples: ① $I_1 = (0, 1)$, $I_2 = (4, 7)$

Is it a cover for: $\{\frac{1}{2}\}$ ✓



$(0, 1)$ ✓, $[0, 1]$ ✗

$(5, 6)$ ✓

$\{4, 5\}$ ✗

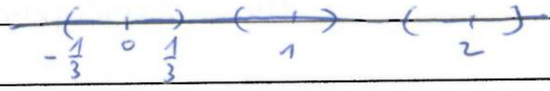
② $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$, $n = 0, 1, 2 \ldots$

Is it a cover for

$\mathbb{N}$ ✓

$\mathbb{Z}$ ✗

$\{-1\}$ ✗



// **Def**: let $S \subset \mathbb{R}$ and $\{I_\alpha\}_{\alpha \in A}$ be a cover of S. A finite collection $\{I_{\alpha_1}, \ldots, I_{\alpha_n}\}$ is called a finite subcover if it itself is a cover for S.

Examples ①: $I_n = (n - \frac{1}{3}, n + \frac{1}{3})$, $n = 0, 1, 2 \ldots$

(a) $S = (-\frac{1}{3}; \frac{1}{3}) \cup \{2\}$

$\{I_n\}$ is a cover for S

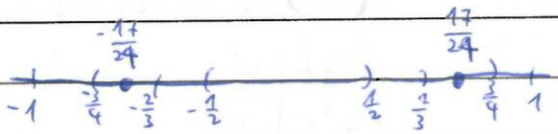
$\{(-\frac{1}{3}, \frac{1}{3}); (2 - \frac{1}{3}; 2 + \frac{1}{3})\}$ is a finite subcover

(b) $S = \mathbb{N}$

$\{I_n\}$ is a cover of $\mathbb{N}$

there is no finite subcover!

② $I_n = (-1 + \frac{1}{n}, 1 - \frac{1}{n})$, $n = 2, 3, 4 \ldots$



(a) $S = [-\frac{17}{24}, \frac{17}{24}]$

$\{I_n\}$ is a cover of S

There is a finite subcover: $(-\frac{3}{4}, \frac{3}{4})$

(b) $S = (-1, 1)$

$\{I_n\}$ is a cover of S

No finite subcover

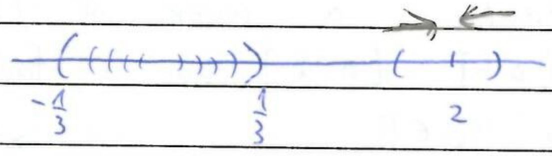
// **Def:** A set $S \subset \mathbb{R}$ is $\boxed{\text{compact}}$ if any cover of S (by open intervals) has a finite subcover.

A set $S \subset \mathbb{R}$ is not compact if there is a cover of S which has <u>no</u> finite subcover.

Examples: (1) $\mathbb{N}$ not compact.

(2) $(-1, 1)$ not compact

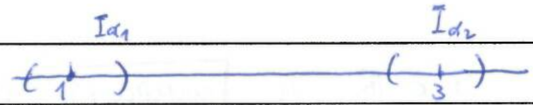
(3) $(-\frac{1}{3}, \frac{1}{3}) \cup \{2\}$ not compact.



(4) $[-\frac{17}{24}, \frac{17}{24}]$ compact

// A is compact $\Leftrightarrow$ every cover of A has finite subcover *

$\underset{\mathbb{R}, \mathbb{R}^2, \mathbb{R}^d}{\Leftrightarrow}$ A is closed and bounded (complex).



$\{1, 3\}$ is compact

Suppose $\{I_\alpha\}_{\alpha \in A}$ is a cover of $\{1, 3\}$

Pick an interval covering 1 and an interval covering 3

They form a finite subcover

// Thm 1.3 (Heine - Borel Theorem)

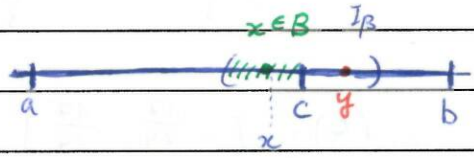
- Every closed interval $[a, b]$ is a compact set *

Proof: Suppose $\{I_\alpha\}_{\alpha \in A}$ is a cover of $[a, b]$

$B = \{ x \in [a, b] : [a, x] \text{ has a finite subcover} \}$

$B \neq \phi$ since $a \in B$

$C = \sup B$



(a) Show $c = b$

(b) $b \in B$

~ (a) Suppose $c < b$

Since $\{I_\alpha\}$ is a cover of $[a, b]$, there is I_β st. $c \in I_\beta$

Since $c = \sup B$ $\exists x \in B \cap I_\beta$

$x \in B$ there is a finite subcover $I_{\alpha_1}, \dots I_{\alpha_m}$ of $[a, x]$

Pick $y \in I_\beta \cap (c, b]$

The interval $[a, y]$ is covered by $I_{\alpha_1}, \dots I_{\alpha_m}, I_\beta$

$\Rightarrow [a, y]$ has a finite subcover so $y \in B$, but $y > c$!

[contradiction]

(b) I_β be an interval covering b

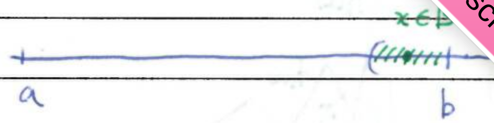
Since $b = \sup B$ (see (a))

$\exists x \in I_\beta \cap [a,b]$

Since $x \in B$ there are finitely many intervals $I_{\alpha_1}, \dots I_{\alpha_m}$ covering $[a,x]$.

But then the intervals $I_{\alpha_1}, \dots, I_{\alpha_m}, I_\beta$ cover $[a,b]$!

So $b \in B$ and $[a,b]$ has a finite subcover $\Rightarrow$ is compact $\square$

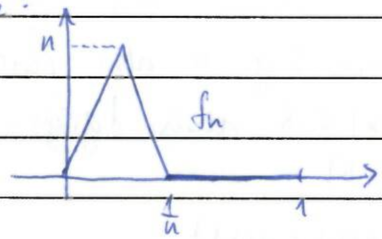


// Thm 1.2 : $\left[\begin{array}{l} f_n \text{ cont} \\ f_n \to f \text{ unif} \Rightarrow f \text{ cont} \end{array}\right]$

f_n cont
f cont $\Rightarrow f_n \to f$ unif (?)
$f_n \to f$ pointwise

NO

Example:



f_n cont.
$f_n \to 0$ pointwise but $f_n \not\to f$ unif.
0 is a cont. function

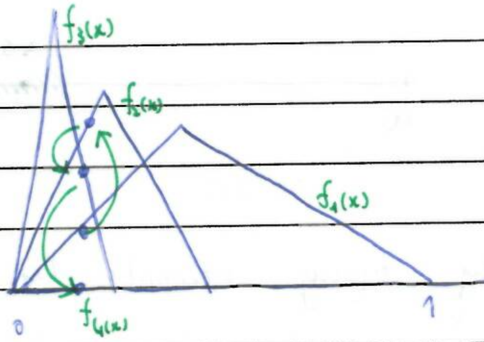
// Thm 1.4 : (Dini's Theorem)

Let $\{f_n\}_{n=1}^{\infty}$, f be real-valued functions on $[a,b]$

Suppose (1) $f_n \to f$ pointwise
(2) all f_n are continuous
(3) f is continuous
(4) $\forall x \in [a,b]$ $\{f_n(x)\}_{n=1}^{\infty}$ is monotone

Then $f_n \to f$ uniformly

eg.



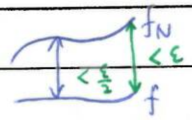
$f_n(x)$ is not monotone $\Rightarrow$ not uniform

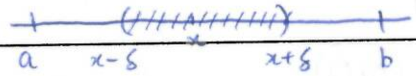
// Proof : Let $\varepsilon > 0$

Let $x \in [a,b]$

since $f_n(x) \to f(x)$ $\exists N(\varepsilon, x)$ st. $\forall n \geq N$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$

In particular, $|f_N(x) - f(x)| < \frac{\varepsilon}{2}$





Denote $g(y) = f_N(y) - f(y)$

Since f_N and f are continuous, g is also continuous.

$\Rightarrow \exists \delta(\varepsilon, x) > 0$ st. if $|y-x| < \delta$, $y \in [a,b]$ then $|g(y) - g(x)| < \frac{\varepsilon}{2}$

$$|f_N(y) - f(y)| = |g(y)| = |g(y) - g(x) + g(x)|$$
$$\leq \underbrace{|g(y) - g(x)|}_{< \varepsilon/2} + \underbrace{|g(x)|}_{< \varepsilon/2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $|y-x| < \delta$
$y \in [a,b]$

By (4) $|f_n(y) - f(y)| < \varepsilon$ $\forall n \geq N$ and $|y-x| < \delta$, $y \in [a,b]$

$I(x) = (x - \delta(x), x + \delta(x))$, $x \in [a,b]$

There is a ~~compact~~ cover of $[a,b]$

Since $[a,b]$ is compact (Heine–Borel Thm)

there is a finite subcover $I(x_1), \ldots, I(x_m)$

Choose $N^*(\varepsilon) = \max_m \{N(\varepsilon, x_1), \ldots\ldots, N(\varepsilon, x_m)\}$

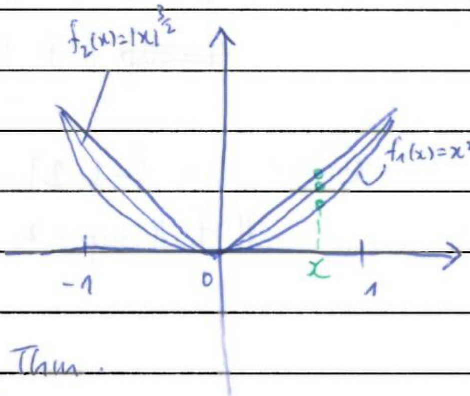
$$\left.\begin{array}{l} y \in [a,b] \Rightarrow y \in I(x_i) \text{ for some } i \\ n \geq N^*(\varepsilon) \Rightarrow n \geq N(\varepsilon, x_i) \end{array}\right\} \Rightarrow |f_n(y) - f(y)| < \varepsilon$$

$\square$

~~$\exists(x)$ proof cont:~~

// Q: $f_n \to f$ unif. $\Rightarrow$ f is ~~continuous~~ diff. ?
all f_n are ~~continuous~~ differentiable.

Example ①: $I = [-1, 1]$
$f_n(x) = |x|^{1+\frac{1}{n}}$ (diff.)
$f(x) = |x|$ (not diff.)



$\forall x$ $|x|^{1+\frac{1}{n}} \to |x|$ (pointwise)
f_n, f are continuous
$f_n(x)$ strictly increasing $\forall x$
$\Big\}$ Dini's Thm.
$f_n \to f$ unif.

②: $f_n \to f$ unif., all f_n diff. $\Rightarrow$ $f_n' \to f'$ (at least pointwise)
f diff.

X (**NO**)

②: $f_n(x) = \frac{1}{\sqrt{n}} \sin(nx)$

$f_n \to 0$ uniformly
$|f_n(x) - 0| = |\frac{1}{\sqrt{n}} \sin(nx)| \le \frac{1}{\sqrt{n}} < \varepsilon$ for n large enough

$f_n'(x) = \frac{1}{\sqrt{n}} n \cos(nx) = \sqrt{n} \cos(nx)$
$f'(x) = 0$
$\Big\}$ look at $x = 0$
$f_n'(0) = \sqrt{n} \cdot 1 \to \infty$

$\Rightarrow$ no pointwise convergence

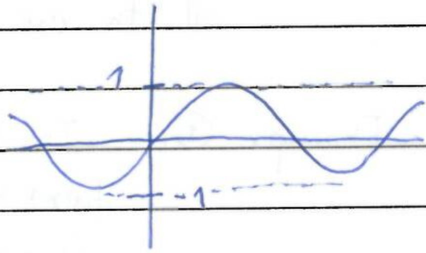
// **Def**: Let $I \subset \mathbb{R}$, $f: I \to \mathbb{R}$

$\|f\|_{sup} \equiv \|f\|_\infty$ is called supremum norm of f

$$\|f\|_{sup} = \sup_{x \in I} |f(x)|$$

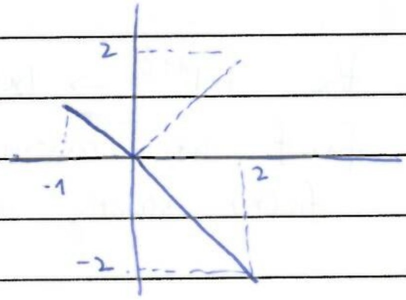
Examples ①: $I = \mathbb{R}$, $f(x) = \sin(x)$

~~sup~~ $\|f\|_{sup} = 1$



②: $I = [-1, 2]$, $f(x) = -x$

$\|f\|_{sup} = 2$



// **Thm 1.5**: Let $I \subset \mathbb{R}$, $\{f_n\}_{n=1}^{\infty}$, $f: I \to \mathbb{R}$

Then $f_n \to f$ uniformly $\Leftrightarrow$

$\|f_n - f\|_{sup} \to 0$ *

Proof: $f_n \to f$ uniformly means

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \underbrace{\forall x \in I \ |f_n(x) - f(x)| < \varepsilon}$

$\Downarrow$

$\|f_n - f\| \leq \varepsilon$

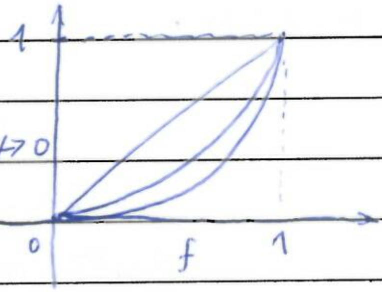
$\Leftrightarrow \|f_n - f\|_{sup} \to 0$ $\square$

Example: ① $I = (0, 1)$, $f_n(x) = x^n$

$f_n \to 0$ pointwise

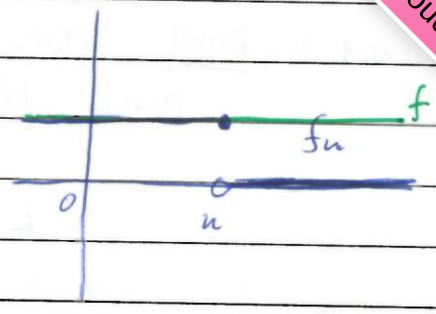
$$\|f_n - f\|_{sup} = \sup_{x \in (0,1)} |x^n - 0| = \sup_{x \in (0,1)} x^n = 1 \not\to 0$$

$\Rightarrow f_n \not\to f$ uniform



② : $I = \mathbb{R}$, f_n

$f_n \to 1$ pointwise



$\| f_n - f \|_{\sup} = \sup_{x \in \mathbb{R}} | f_n(x) - f(x) | = 1 \not\to 0$

takes two values: 0,1

// Thm 1.6 :

Let $\{f_n\}$, f be real-valued functions on $[a,b]$

Suppose $f_n \longrightarrow f$ uniformly (all f_n are Riemann-integrable)

Then f is Riemann-integrable

$$\left[\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx \right] *$$

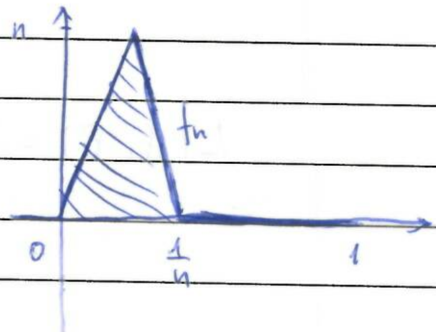
Q : $f_n \to f$ pointwise, all f_n are integrable $\Big\} \Rightarrow$ f is integrable and $\int_a^b f_n(x) \, dx \to \int_a^b f(x) \, dx$ ⓧ

Example : ① $f_n \to 0$ pointwise

The limit function is integrable

but $\int_0^1 f_n(x) \, dx = n \cdot \frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2}$

$\int_0^1 0 \, dx = 0$ X



② : Example where f is not even integrable - see hw.

~ Proof : Recall : Take a partition $P = \{a = t_0 < t_1 < \dots < t_n = b\}$

Define $U(f,P) = \sum_{i=1}^{n} \sup_{x\in[t_{i-1},t_i]} f(x) . (t_i - t_{i-1})$

$L(f,P) = \sum_{i=1}^{n} \inf_{x\in[t_{i-1},t_i]} f(x) \cdot (t_i - t_{i-1})$

f is integrable if $\forall \varepsilon > 0 \; \exists P$ st. $U(f,P) - L(f,P) < \varepsilon$

Pick $\varepsilon > 0$, Since $f_n \to f$ uniformly

$\| f_n - f \|_{sup} \to 0$

So $\exists n$ st. $\| f_n - f \|_{sup} < \frac{\varepsilon}{4(b-a)}$

This means

$f_n(x) - \frac{\varepsilon}{4(b-a)} < f(x) < f_n(x) + \frac{\varepsilon}{4(b-a)} \quad \forall x \in [a,b]$

Since f_n is integrable $\exists P$ st. $U(f_n,P) - L(f_n,P) < \frac{\varepsilon}{2}$

$$U(f,P) = \sum \sup_{x\in[t_{i-1},t_i]} \underbrace{f(x)}_{< f_n(x) + \frac{\varepsilon}{4(b-a)}} (t_i - t_{i-1})$$

$$< U(f_n,P) + \frac{\varepsilon}{4(b-a)} \cdot (b-a)$$

$$= U(f_n,P) + \frac{\varepsilon}{4}$$

$$L(f,P) = \sum \inf_{x\in[t_{i-1},t_i]} f(x)(t_i - t_{i-1})$$

$$> L(f_n,P) - \frac{\varepsilon}{4}$$

$$U(f,P) - L(f,P) < \underbrace{U(f_n,P) + \frac{\varepsilon}{4} - L(f_n,P)}_{< \frac{\varepsilon}{2}} + \frac{\varepsilon}{4} < \varepsilon$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b (f_n(x) - f(x))\,dx \right|$$

$$\leq \int_a^b \underbrace{|f_n(x) - f(x)|}_{\leq \| f_n - f \|_{sup}}\,dx$$

(No x !! just numbers)

$$\leq \| f_n - f \|_{sup} \cdot \int_a^b dx = (b-a) \underbrace{\| f_n - f \|_{sup}}_{\downarrow 0}$$

$$\Rightarrow \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx \quad \square$$

// **Def:** Let $I \subset \mathbb{R}$, $\{g_n\}_{n=1}^{\infty}$ be real-valued functions on I
$\sum_{n=1}^{\infty} g_n$ is called a series of functions.

$f_n(x) = \sum_{i=1}^{n} g_i(x)$ is called the n-th <u>partial sum</u>.

- We say the series $\sum_{n=1}^{\infty} g_n$ converges pointwise on I if f_n converges pointwise to some function f
- f is called the sum of the series
- We say the series $\sum_{n=1}^{\infty} g_n$ converge uniformly if f_n converge uniformly.

Example: $\sum_{n=0}^{\infty} x^n$

- pointwise ?
- the sum ?
- uniformly ?

2 cases: $I = [0, \frac{1}{2}]$; $I = [0, 1)$

$$f_n(x) = \sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$$

partial sum; $x^{n+1} \to 0$

f_n converge pointwise to $f(x) = \frac{1}{1-x}$
$\Rightarrow \sum_{n=0}^{\infty} x^n$ converges pointwise on both domains

and the sum is $\frac{1}{1-x}$

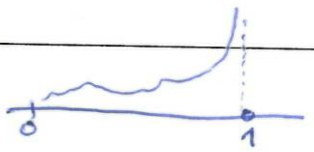
$$\| f_n - f \|_{\sup} = \sup \left| \frac{x^{n+1} - 1}{x - 1} - \frac{1}{1-x} \right|$$

$$= \sup \left| \frac{x^{n+1}}{x-1} \right|$$

- on $[0, \frac{1}{2}]$: $\| f_n - f \|_{\sup} = \sup_{[0,\frac{1}{2}]} \left| \frac{x^{n+1}}{x-1} \right| \leq \frac{\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}} = \frac{1}{2^n} \xrightarrow{n \to \infty} 0$

The series converges uniformly on $[0, \frac{1}{2}]$ / $x \in [0, \frac{1}{2}]$, $1-x \in [\frac{1}{2}, 1]$

- on $[0,1)$ $\| f_n - f \|_{\sup} = \sup_{[0,1)} \left| \frac{x^{n+1}}{x-1} \right| = \infty \not\to 0$



Series doesn't converge uniformly on $[0,1)$

// $\sum_{n=1}^{\infty} g_n$ series of functions

- $\sum_{n=1}^{\infty} g_n$ converges pointwise $\Leftrightarrow$ $f_n = \sum_{i=1}^{n} g_i$ converges pointwise
- " uniformly $\Leftrightarrow$ " uniformly.

// **Def:** Let $I \subset \mathbb{R}$, $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions on I. We say that $\{f_n\}_{n=1}^{\infty}$ is a uniform Cauchy sequence if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n,m \geq N$, $\|f_n - f_m\|_{\sup} < \varepsilon$

// **Thm 1.7:** $\{f_n\}_{n=1}^{\infty}$ converges uniformly $\Leftrightarrow$ it is a uniform Cauchy sequence.

[Central principle of uniform convergence - CPUC]

Proof: Suppose $\{f_n\}$ converges uniformly to some f

$\forall \varepsilon > 0 \quad \exists N \quad \forall n \geq N \quad \forall x \in I \quad |f_n(x) - f(x)| < \frac{\varepsilon}{4}$

Then $\forall n,m \geq N \quad \forall x \in I \quad |f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m(x)|$

$$\leq \underbrace{|f_n(x) - f(x)|}_{< \frac{\varepsilon}{4}} + \underbrace{|f_m(x) - f(x)|}_{< \frac{\varepsilon}{4}} < \frac{\varepsilon}{2}$$

$$\|f_n - f_m\|_{\sup} \leq \frac{\varepsilon}{2} < \varepsilon$$

Suppose $\{f_n\}$ is uniform Cauchy

$\forall \varepsilon > 0 \quad \exists N \quad \forall n,m \geq N \quad \|f_n - f_m\|_{\sup} < \varepsilon$

Let $x \in I \Rightarrow \forall n,m \geq N \quad |f_n(x) - f_m(x)| < \varepsilon \quad (*)$

$\{f_n(x)\}_{n=1}^{\infty}$ - Cauchy sequence of numbers

sequence of numbers $\Rightarrow$ converges to some limit $f(x)$

Let $m \to \infty$ in $(*)$ $\Rightarrow |f_n(x) - f(x)| \leq \varepsilon$

$\Rightarrow f_n \to f$ uniformly

$\square$

// Thm 1.8 (Weierstrass M - test)

Let $I \subset \mathbb{R}$, let $\{g_n\}_{n=1}^{\infty}$ be real-valued functions on I

Let $\sum_{n=1}^{\infty} M_n$ be a series of numbers st.

- $|g_n(x)| \leq M_n \quad \forall x \in I$
- $\sum_{n=1}^{\infty} M_n < \infty$

Then $\sum_{n=1}^{\infty} g_n$ converges uniformly. *

Proof: Let $\varepsilon > 0$

$\sum_{i=1}^{\infty} M_i < \infty \Rightarrow \left\{ \sum_{i=1}^{n} M_i \right\}_{n=1}^{\infty}$ converges $\Rightarrow$ it is a Cauchy sequence.

(sequence of partial sums)

$\exists N \quad \forall n, m \geq N, \ \left| \sum_{i=1}^{n} M_i - \sum_{i=1}^{m} M_i \right| < \frac{\varepsilon}{2} \Rightarrow \sum_{i=m+1}^{n} M_i < \frac{\varepsilon}{2}$

Look $\left\{ \sum_{i=1}^{n} g_i \right\}_{i=1}^{\infty}$, take $n > m \geq N$

$$\left| \sum_{i=1}^{n} g_i(x) - \sum_{i=1}^{m} g_i(x) \right| = \left| \sum_{i=m+1}^{n} g_i(x) \right| \leq \sum_{i=m+1}^{n} \underbrace{|g_i(x)|}_{\leq M_i} \leq \sum_{i=m+1}^{n} M_i < \frac{\varepsilon}{2}$$

$$\left\| \sum_{i=1}^{n} g_i - \sum_{i=1}^{m} g_i \right\|_{\sup} \leq \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow \left\{ \sum_{i=1}^{n} g_i \right\}_{n=1}^{\infty}$ is uniform Cauchy

CPUC $\Rightarrow$ it converges uniformly

$\Rightarrow$ the series converges uniformly $\square$

Examples: Do the series converge pointwise / uniformly?

① $\sum_{n=1}^{\infty} \frac{\sin(x)}{2^n}$, $\left|\frac{\sin(x)}{2^n}\right| \le \frac{1}{2^n}$ — doesn't depend on x.

$\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$

Converges uniformly

② $\sum_{n=1}^{\infty} \frac{1}{n^2+x}$ on $[0,\infty)$, $\frac{1}{n^2+x} \le \frac{1}{n^2}$ $\forall x \in [0,\infty)$

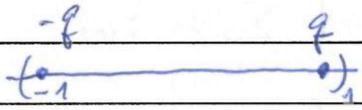
$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

Converges uniformly

③ $\sum_{n=1}^{\infty} x^n$

$[0.999, 0.999]$

(a) $[-q, q]$, $0 < q < 1$,



$|x^n| \le q^n$, $\sum q^n < \infty$

$\Rightarrow$ by the M-test, the series converges uniformly.

(b) $(-1, 1)$

$f_n(x) = \sum_{i=1}^{n} x^i$

$f_{n+1}(x) - f_n(x) = x^{n+1}$ | $\forall \varepsilon > 0, \exists N$ $\forall n,m \ge N$, $\|f_m - f_n\|_{sup} < \varepsilon$

$\|f_{n+1} - f_n\|_{sup} = \sup_{x\in(-1,1)} |x^{n+1}| = 1 \not\to 0$

So f_n is not uniform Cauchy.

26/1/20

// $\sum g_n$

*** (1) To prove unif. conv. use M-test.
(2) To disprove unif conv. use CPUC

Example: (4) $\sum_{n=1}^{\infty} (\sin(x))^n$, $[0, \frac{\pi}{2})$ and $(0, \frac{\pi}{4})$

pointwise : $0 < \sin x < 1$

$\sum (\sin(x))^n$ converges $\forall x$ (= pointwise) on both domains.

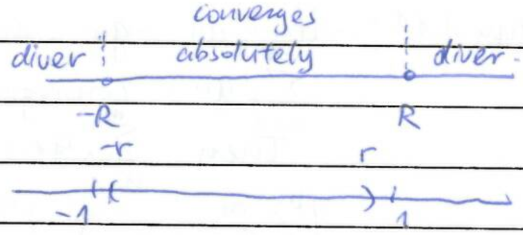
uniform : $(0, \frac{\pi}{4})$

$0 < |(\sin x)^n| < \left(\frac{\sqrt{2}}{2}\right)^n$; $\sum \left(\frac{\sqrt{2}}{2}\right)^n < \infty$

M-test $\Rightarrow$ uniform convergence.

Power series and uniform convergence.

$\sum_{n=0}^{\infty} a_n x^n$



Example $\sum_{n=0}^{\infty} x^n$, $R = 1$

does not converge uniformly on $(0,1)$ and on $(-1,1)$
does converge uniformly on $(0, \frac{1}{2})$ and $(-r, r)$ with $0 < r < 1$

$(0, \frac{\pi}{2})$ $\| f_{n+1} - f_n \|_{\sup} = \sup_{x \in (0, \frac{\pi}{2})} |(\sin(x))^{n+1}| = 1 \not\to 0$

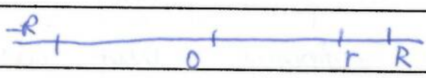
partial sums.

$\Rightarrow$ no uniform conv.

// Thm 1.9: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with r.o.c = R
let $0 < r < R$, Then the power series converges on $[-r, r]$ uniformly.

Proof: $|a_n x^n| \leq |a_n r^n|$ for all $x \in [-r, r]$

$\sum_{n=1}^{\infty} |a_n r^n| < \infty$



(due to the abs. convergence at r)

By the M-test the power series converges absolutely.

// Thm 1.10: If all g_n are continuous and $\sum_{n=1}^{\infty} g_n$ converges uniformly.

Then the sum $\sum_{n=1}^{\infty} g_n$ is continuous.

Proof: $\underbrace{\sum_{i=1}^{n} \underset{\uparrow \text{ cont.}}{g_i}}_{\text{cont.}} \longrightarrow \sum_{i=1}^{\infty} g_i$ uniformly.

$\Rightarrow$ cont. (Thm about unif. conv + cont.) (1.2) $\square$

// Thm 1.11: If all g_n are integrable on $[a,b]$. and $\sum_{n=1}^{\infty} g_n$ converges uniformly

Then $\sum_{n=1}^{\infty} g_n$ is integrable and

$$\int_a^b \left(\sum_{n=1}^{\infty} g_n(x) \right) dx = \sum_{n=1}^{\infty} \int_a^b g_n(x)\, dx$$

Proof: $\underbrace{\sum_{i=1}^{n} \underset{\uparrow \text{ integrable}}{g_i}}_{\text{integrable}} \rightarrow \sum_{i=1}^{\infty} g_i$ uniformly.

$\underset{\text{Thm 1.6}}{\Rightarrow} \sum_{i=1}^{\infty} g_i$ is integrable.

$$\int_a^b \left(\sum_{i=1}^{n} g_i(x) \right) dx \longrightarrow \int_a^b \left(\sum_{i=1}^{\infty} g_i(x) \right) dx$$

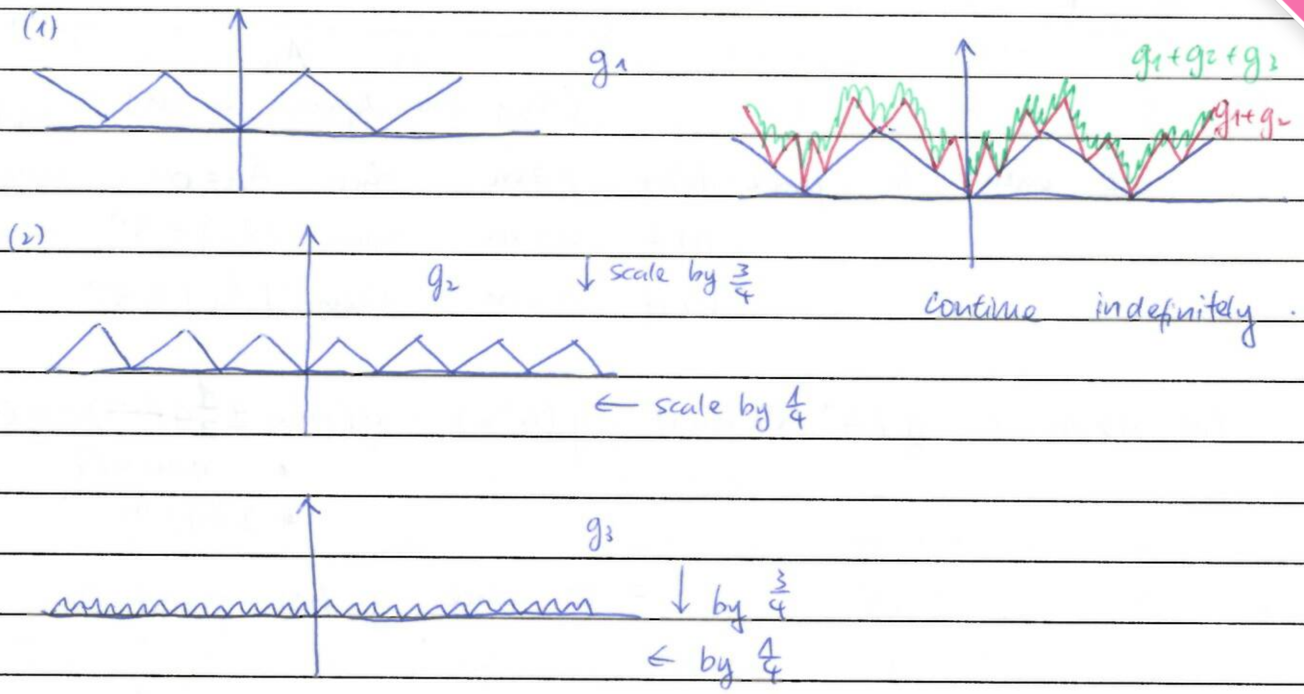
$$\sum_{i=1}^{n} \int_a^b g_i(x)\, dx \longrightarrow \text{---}\,{''}\,\text{---}$$

$$\sum_{i=1}^{\infty} \int_a^b g_i(x)\, dx = \int_a^b \left(\sum_{i=1}^{\infty} g_i(x) \right) dx$$

$\square$

// Continuous but ~~continuous~~ nowhere differentiable function

Idea:



// Thm 1.12: There exists a continuous nowhere differentiable function on $\mathbb{R}$

Proof: ~ let $g(x) = |x|$ for $x \in [-1,1]$

and extend it 2 periodically to $\mathbb{R}$

Define $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x)$

scale factor

$$\left|\left(\frac{3}{4}\right)^n g(4^n x)\right| \le \left(\frac{3}{4}\right)^n \quad ; \quad \sum \left(\frac{3}{4}\right)^n < \infty$$

By the M-test, the series converges uniformly

$\rightarrow$ f is well-defined $\forall x$

Each $\left(\frac{3}{4}\right)^n g(4^n x)$ is continuous

The series converges uniformly $\xrightarrow{\text{Thm 1.10}}$ f is continuous.

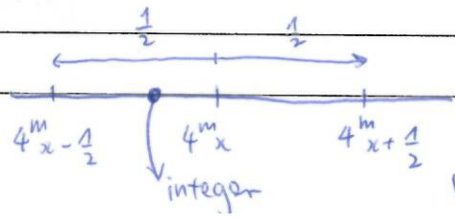
~ Prove f is nowhere differentiable

let $x \in \mathbb{R}$, want to show $\frac{f(x+h)-f(x)}{h}$ has no limit as $h \to 0$

Construct $h_m \to 0$ st. $\left|\frac{f(x+h_m)-f(x)}{h_m}\right| \to \infty$

$h_m = \pm 4^{-m}$, where

$h_m \to 0$



(+) if there is no integer in $(4^m x, 4^m x + \frac{1}{2})$

(−) if there is no integer in $(4^m x - \frac{1}{2}, 4^m x)$.

$$\left|\frac{f(x+h_m)-f(x)}{h_m}\right| = \left|\sum_{n=0}^{\infty}\underbrace{\left(\frac{3}{4}\right)^n\left(\frac{g(4^n(x+h_m))-g(4^n x)}{h_m}\right)}_{A_n}\right|$$

(they also depend on m)

We want to prove: (a) if $n \geq m$ then $A_n = 0$

(b) if $n = m$ then $|A_n| = 3^n$

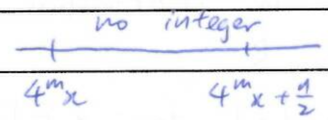
(c) if $n < m$, then $|A_n| \leq 3^n$

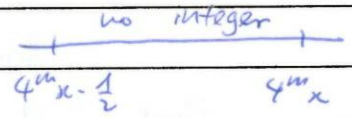
(a) $n > m$: $g(4^n(x+h_m)) - g(4^n x) = g(4^n x \pm \frac{1}{2}4^{n-m}) - g(4^n x)$

- $n-m \in \mathbb{N}$
- $2 \mid 4^{n-m}$

$= 0$ since g has period 2

(b) $n = m$, $|g(4^n(x+h_m)) - g(4^n x)| = |g(4^m x \pm \frac{1}{2}) - g(4^m x)|$

(+) 

(−) 

$= |4^m x \pm \frac{1}{2} - 4^m x| = \frac{1}{2}$

$$|A_n| = \left|\left(\frac{3}{4}\right)^n \frac{\frac{1}{2}}{\pm\frac{1}{2}\cdot 4^{-n}}\right| = 3^n$$

(c) $n < m$ $|g(4^n(x+h_m)) - g(4^n x)| \leq |4^n x \pm \frac{1}{2}\cdot 4^{n-m} - 4^n x|$

$|g(a) - g(b)| \leq |a-b|$

$= \frac{1}{2}4^{n-m}$

$$|A_n| \leq \left|\left(\frac{3}{4}\right)^n \cdot \frac{\frac{1}{2}\cdot 4^{n-m}}{\pm\frac{1}{2}4^{-m}}\right| = 3^n$$

$\sim$ $\left|\frac{f(x+h_m)-f(x)}{h_m}\right| \underset{(a)}{=} |A_0 + A_1 + \dots + A_m| \geq |A_m| - |A_0 + \dots + A_{m-1}|$

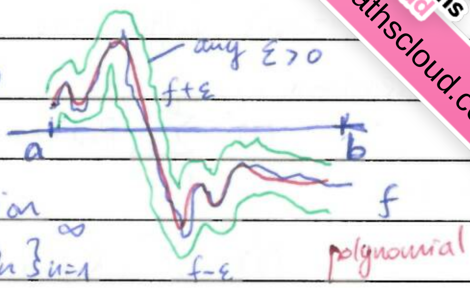
$|a+b| \geq |a| - |b|$

$\geq |A_m| - |A_0| - |A_1| - \dots - |A_{m-1}|$

$$\underset{(b)+(c)}{\geq} 3^m - \sum_{n=0}^{m-1} 3^n = 3^m - \frac{3^m - 1}{3-1} = \frac{3^m}{2} + \frac{1}{2} \to \infty$$

$\square$

// Weierstrass Approximation Theorem (Thm 1.13)



Let $f: [a,b] \to \mathbb{R}$ be a continuous function. There is a sequence of polynomials $\{p_n\}_{n=1}^{\infty}$ which converges to f uniformly on $[a,b]$.

Proof: later!

~ Discuss $[0,1]$

Let $f: [0,1] \to \mathbb{R}$ be a cont. function.

Take $x \in [0,1]$, take an unfair coin: Head $\to$ w.p. x

Tail $\to$ w.p. $1-x$

Toss it n-times, count the number of heads: Y_n

$Y_n \approx nx \iff \frac{Y_n}{n} \approx x$

$$f\left(\frac{Y_n}{n}\right) \approx f(x), \quad \underbrace{E\, f\left(\frac{Y_n}{n}\right)}_{\text{non-random}} \approx f(x) \quad \text{as } n \to \infty$$

$$\| $$

$$\sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot P(Y_n = k) = \underbrace{\sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}}_{\text{happens to a polynomial.!}}$$

Def: $p_{kn}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $n = 0, 1, 2, \ldots$
$0 \le k \le n$

Def: Let $f: [0,1] \to \mathbb{R}$ be a function

$$B_n^f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{kn}(x) \quad \text{[Bernstein's polynomial corr. to } f\text{]}$$

$n = 0, 1, 2, 3, \ldots$

Bernstein's Polynomials

// Lemma

(1) $\sum_{k=0}^{n} p_{kn}(x) = 1$

(2) $\sum_{k=0}^{n} k\, p_{kn}(x) = nx$ ← expectation

(3) $\sum_{k=0}^{n} (k-nx)^2 p_{kn}(x) = nx(1-x)$

$\forall n = 0,1,2,\dots$
$\forall x \in [0,1]$

Proof: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

(1) $\sum_{k=0}^{n} p_{kn}(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x+1-x)^n = 1$

(2) $k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = n \cdot \frac{(n-1)!}{(k-1)!((n-1)-(k-1))}$

$= n \cdot \binom{n-1}{k-1}$

$$\sum_{k=0}^{n} k\, p_{kn}(x) = \sum_{k=1}^{n} \underbrace{k\binom{n}{k}}_{n\binom{n-1}{k-1}} x^k (1-x)^{n-k}$$

$$= nx \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{(n-1)-(k-1)}$$

$$= nx \sum_{i=0}^{n-1} \binom{n-1}{i} x^i (1-x)^{n-1-i} = nx(x+1-x)^{n-1} = nx$$

(3) $k(k-1)\binom{n}{k} = k(k-1)\frac{n!}{k!(n-k)!} = n(n-1)\frac{(n-2)!}{(k-2)!((n-2)-(k-2))}$

$= n(n-1)\binom{n-2}{k-2}$

$$\sum_{k=0}^{n} k(k-1)p_{kn}(x) = \sum_{k=2}^{n} \underbrace{k(k-1)\cdot\binom{n}{k}}_{n(n-1)\binom{n-2}{k-2}} x^k (1-x)^{n-k}$$

~~$\sum_{k=0}^{n} (k-nx)^2 p_{kn}(x)$~~

~~$k^2 - 2knx + n^2x^2$~~

~~$k(k-1) + k - 2knx + n^2x^2$~~

~~$= \sum_{k=0}^{n}$~~

$$= n(n-1)x^2 \sum_{k=2}^{n} \binom{n-2}{k-2} x^{k-2}(1-x)^{(n-2)-(k-2)}$$

$$= n(n-1)x^2 \sum_{i=0}^{n-2} \binom{n-2}{i} x^i (1-x)^{n-2-i}$$

$$= n(n-1)x^2(x+1-x)^{n-2}$$

$$= n(n-1)x^2$$

⇒

$$\sum_{k=0}^{n} \underbrace{(k-nx)^2}_{\substack{k^2-2knx+n^2x^2 \\ k(k-1)+k-2knx+n^2x^2}} p_{kn}(x) = \overbrace{\sum_{k=0}^{n} k(k-1)\,p_{kn}(x)}^{n(n-1)x^2} + \overbrace{\sum_{k=0}^{n} k\,p_{kn}(x)}^{nx} - 2nx\overbrace{\sum_{k=0}^{n} k\,p_{kn}(x)}^{nx} + n^2x^2 \sum_{k=0}^{n} p_{kn}(x) = 1$$

$$= n(n-1)x^2 + nx - 2n^2x^2 + n^2x^2 = -nx^2 + nx = nx(1-x)$$

// Recall: $f : [0,1] \to \mathbb{R}$

f is called uniformly continuous if

- $\forall \varepsilon > 0 \;\; \exists \delta > 0$ s.t. if $|y-x| < \delta$, $x, y \in [0,1]$, then $|f(x) - f(y)| < \varepsilon$

Normally: unif. continuity $\Rightarrow$ continuity.

- On $[0,1]$ unif. continuity $\Leftrightarrow$ continuity (any $[a,b]$)

// Thm 1.14 (Weierstrass Approximation Thm on $[0,1]$)

Let $f : [0,1] \to \mathbb{R}$ be a cont. function.

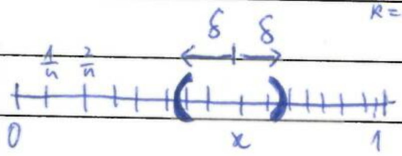
Then $B_n^f \to f$ uniformly on $[0,1]$.

Proof: Let $\varepsilon > 0$

$$|B_n^f(x) - f(x)| = \left| \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) p_{kn}(x) - \sum_{k=0}^{n} f(x)\, p_{kn}(x) \right|$$

$\cdot \; \sum_{k=0}^{n} p_{kn}(x) = 1$

$$= \left| \sum_{k=0}^{n} \left(f\left(\tfrac{k}{n}\right) - f(x) \right) p_{kn}(x) \right|$$



$$\leq \sum_{k=0}^{n} \left| f\left(\tfrac{k}{n}\right) - f(x) \right| p_{kn}(x)$$

f is continuous on $[0,1]$ $\Rightarrow$ f is uniformly continuous on $[0,1]$

$\Rightarrow \exists \delta > 0$ s.t. if $|y-x| < \delta$, $y, x \in [0,1]$, then $|f(y) - f(x)| < \frac{\varepsilon}{2}$

$$= \sum_{k : |\frac{k}{n} - x| < \delta} \underbrace{\left| f\left(\tfrac{k}{n}\right) - f(x) \right|}_{< \frac{\varepsilon}{2}} p_{kn}(x) + \sum_{k : |\frac{k}{n} - x| \geq \delta} \underbrace{\left| f\left(\tfrac{k}{n}\right) - f(x) \right|}_{\leq |f(\frac{k}{n})| + |f(x)| \leq 2\|f\|_{\sup}} p_{kn}(x)$$

$$< \frac{\varepsilon}{2} \sum_{k : |\frac{k}{n} - x| < \delta} p_{kn}(x) + 2\|f\|_{\sup} \cdot \sum_{k : |\frac{k}{n} - x| \geq \delta} p_{kn}(x)$$

$\Rightarrow$

(cont): $< \frac{\varepsilon}{2} \underbrace{\sum_{k:\left|\frac{k}{n}-x\right|<\delta} p_{kn}(x)}_{\leq \sum_{k=0}^{n} p_{kn}(x) = 1} + 2\|f\|_{sup} \underbrace{\sum_{k:\left|\frac{k}{n}-x\right|\geq\delta} p_{kn}(x)}_{\left|\frac{k}{n}-x\right| \geq \delta}$ 1

$$\frac{(k-nx)^2}{n^2} \geq \delta^2 \iff \boxed{\frac{(k-nx)^2}{n^2\delta^2} \geq 1}$$

$$< \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{n^2\delta^2} \sum_{k:\left|\frac{k}{n}-x\right|\geq\delta} (k-nx)^2 p_{kn}(x)$$

$$< \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{n^2\delta^2} \underbrace{xn(1-x)}_{\leq 1}$$

$$\leq \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{n}$$

Choose N so that $\frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{N} < \frac{\varepsilon}{2}$

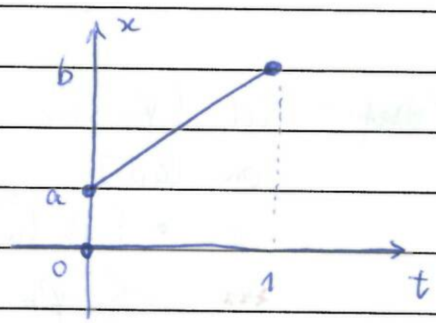
Then $\forall\, n \geq N$

$$|B_n^f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{n}$$

$$\leq \frac{\varepsilon}{2} + \frac{2\|f\|_{sup}}{\delta^2} \cdot \frac{1}{N} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \square$$

// Thm 1.13 (WAT)
let $f:[a,b]\to\mathbb{R}$ be a continuous function. Then $\exists$ a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ which converges to f uniformly on $[a,b]$.

Proof: $[0,1] \xrightarrow[x(t)=a+t(b-a)]{} [a,b] \xrightarrow{f} \mathbb{R}$



$g(t) = f(x(t))$
$g:[0,1]\to\mathbb{R}$ is continuous

$\Rightarrow$ Thm 1.4: $B_n^g \to g$ unif on $[0,1]$
Define $P_n(x) = B_n^g(t(x)) = B_n^g\left(\frac{x-a}{b-a}\right) \leftarrow$ polynomials.

$$\|f-P_n\|_{sup} = \sup_{x\in[a,b]}\left|f(x) - B_n^g\left(\frac{x-a}{b-a}\right)\right| = \sup_{x\in[0,1]}\Big|\underbrace{f(x(t))}_{g(t)} - B_n^g\Big(\underbrace{\frac{x(t)-a}{b-a}}_{t}\Big)\Big|$$

$$= \|g-B_n^g\|_{sup} \to 0 \Rightarrow P_n \to f \text{ unif on } [a,b] \quad \square$$

// < Fourier Series >

notation : $R[a,b]$ - the set of Riemann-integrable functions on $[a,b]$

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx \quad \forall f,g \in R[a,b].$$

$$\|f\|_2 = \sqrt{\langle f,f\rangle} = \sqrt{\int_a^b f(x)^2\,dx} \quad \text{- 2 norm}$$

// **Def** : Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a,b]$.

- $\{\varphi_n\}_{n=1}^{\infty}$ is an ~~orthonormal~~ orthogonal system if
  *** $\langle \varphi_n, \varphi_m\rangle = 0 \quad \forall m \neq n$
- $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal system (o.n.s) if it is ① orthogonal and ② $\|\varphi_n\|_2 = 1 \ \forall n$
  $\sqrt{\langle \varphi_n, \varphi_n\rangle}$, ie $\langle \varphi_n, \varphi_n\rangle = 1$

Example : ① Trigonometric ons : $\left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx), n \in \mathbb{N}\right\}$ on $[-\pi,\pi]$

Check : $\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{2\pi}}\right)^2 dx = 1$

$$\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{\pi}}\cos(nx)\right)^2 dx = 1 \qquad \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{\pi}}\cos(nx)\,dx = 0$$

$$\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{\pi}}\sin(nx)\right)^2 dx = 1 \qquad \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{\pi}}\sin(nx)\,dx = 0$$

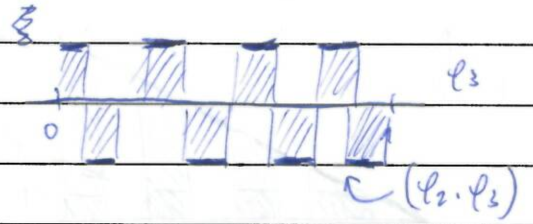
$$\int_{-\pi}^{\pi}\frac{1}{\sqrt{\pi}}\cos(nx)\frac{1}{\sqrt{\pi}}\cos(nx)\,dx = 0 \text{ if } m\neq n, \qquad \int_{-\pi}^{\pi}\frac{1}{\sqrt{\pi}}\cos(nx)\frac{1}{\sqrt{\pi}}\sin(nx)\,dx = 0$$

$$\int_{-\pi}^{\pi}\frac{1}{\sqrt{\pi}}\sin(nx)\frac{1}{\sqrt{\pi}}\sin(nx)\,dx = 0 \text{ if } m\neq n.$$

② : $[0,1]$

etc. $\langle \varphi_n, \varphi_n \rangle = \int_0^1 \underbrace{\varphi_n(x)^2}_{1} \, dx = 1 \quad \forall n.$

$\langle \varphi_2, \varphi_3 \rangle = \int_0^1 \varphi_2(x)\varphi_3(x)\, dx$ true for any $m \neq n$



// **Def**: let $f \in R[a,b]$ and $\{\varphi_n\}_{n=1}^{\infty}$ be an o.n.s on $[a,b]$

$$a_n = \langle f, \varphi_n \rangle \equiv \int_a^b f(x)\varphi_n(x)\, dx, \quad n \in \mathbb{N}$$

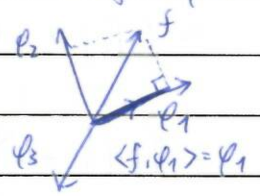
are Fourier ~~series~~ coefficients of f with respect to $\{\varphi_n\}$

$\sum_{n=1}^{\infty} a_n \varphi_n$ is ~~an~~ the Fourier series of f wrt. $\{\varphi_n\}$.

Given $\{\varphi_n\}$

$$f \xrightarrow[\text{associate}]{} \sum_{n=1}^{\infty} a_n \varphi_n$$

- this series can diverge!
- in particular, $f(x) \neq \sum_{n=1}^{\infty} a_n\varphi_n$ in general
- will prove if f is differentiable then $f(x) = \sum_{n=1}^{\infty} a_n\varphi_n$
- we wont prove (but its true) if f is continuous then ~~f(x)~~ $f(x) = \sum_{n=1}^{\infty} a_n \varphi_n$.
- If $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}^3$ (orthonormal)

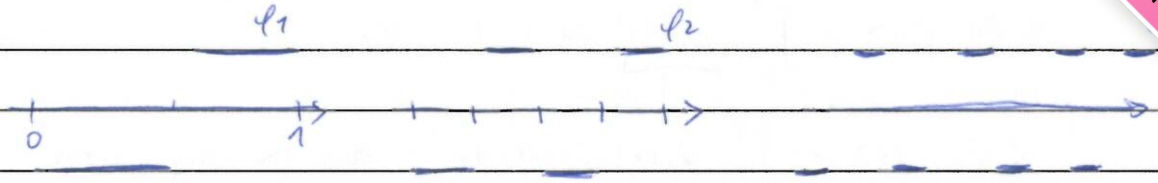


$$f = \langle f, \varphi_1 \rangle \cdot \varphi_1$$
$$= \langle f, \varphi_2 \rangle \cdot \varphi_2$$
$$= \langle f, \varphi_3 \rangle \cdot \varphi_3$$

a_1, a_2, a_3

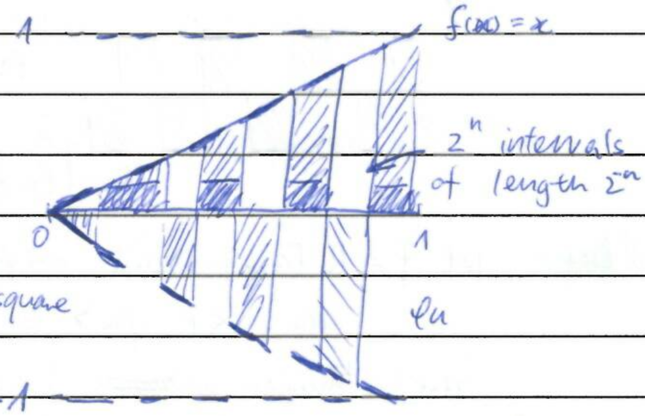
$$= a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3$$

Example:



$f(x) = x$

$a_n = \int_0^1 x \, \varphi_n(x) \, dn$



$$a_n = \int_0^1 x \, \varphi_n(x) \, dx$$

$$= \underset{\substack{\uparrow \\ \text{number of squares}}}{2^{n-1}} \cdot 2^{-2n} \leftarrow \text{area of each square}$$

$$= \frac{1}{2^{n+1}}$$

$\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \varphi_n$ is the Fourier Series of $f(x) = x$ wrt $\{\varphi_n\}$

// **Theorem 2.1**: (Least Squares approximation)
Let $f \in R[a,b]$, $\{\varphi_n\}$ be an ons; $\{a_n\}$ be the Fourier coefficients. Then

** $\left\| f - \sum_{i=1}^{n} a_i \varphi_i \right\|_2 \le \left\| f - \sum_{i=1}^{n} c_i \varphi_i \right\|_2$, for any n and $c_1, \dots, c_n \in \mathbb{R}$ with the equality if and only if $c_1 = a_1, \dots, c_n = a_n$.

Proof: $\left\| f - \sum_{i=1}^{n} a_i \varphi_i \right\|_2^2 = \langle f - \sum_{i=1}^{n} a_i \varphi_i \, ; \, f - \sum_{j=1}^{n} a_j \varphi_j \rangle$

RHS. $= \langle f, f \rangle - \langle f, \sum_{j=1}^{n} a_j \varphi_j \rangle - \langle \sum_{i=1}^{n} a_i \varphi_i, f \rangle + \langle \sum_{i=1}^{n} a_i \varphi_i, \sum_{j=1}^{n} a_j \varphi_j \rangle$

$$= \|f\|_2^2 - 2 \sum_{i=1}^{n} a_i \underbrace{\langle f, \varphi_i \rangle}_{a_i} + \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \underbrace{\langle \varphi_i, \varphi_j \rangle}_{\delta_{ij}}$$

$$= \|f\|_2^2 - 2 \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i^2 = \|f\|_2^2 - \sum_{i=1}^{n} a_i^2$$

LHS: $\| f - \sum_{i=1}^{n} a_i \varphi_i \|_2^2$ = 2 computations are the same as above with a's replaced by c's.

$$= \|f\|_2^2 - 2\sum_{i=1}^{n} c_i \underbrace{\langle f, \varphi_i \rangle}_{a_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \underbrace{\langle \varphi_i, \varphi_j \rangle}_{\delta_{ij}}$$

$$= \|f\|_2^2 - 2\sum_{i=1}^{n} c_i a_i + \sum_{i=1}^{n} c_i^2$$

$$= \|f\|_2^2 - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} (a_i^2 - 2a_i c_i + c_i^2)$$

$$= \|f\|_2^2 - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} (a_i - c_i)^2$$

so $\| f - \sum_{i=1}^{n} a_i \varphi_i \|_2^2 \leq \| f - \sum_{i=1}^{n} c_i \varphi_i \|^2$

"=" $\Leftrightarrow a_1 = c_1, \dots a_n = c_n$

$\Rightarrow \| \dots \|_2 \leq \| \dots \|_2$ ← without squares □

// Thm 2.2 (Bessel's inequality):

$$\boxed{\begin{array}{l} \sum_{n=1}^{\infty} a_n^2 \leq \|f\|_2^2 \\ \text{In particular, } a_n \to 0 \text{ as } n \to \infty \end{array}}$$

Proof: Look at the previous proof.

$$\underbrace{\| f - \sum_{i=1}^{n} a_i \varphi_i \|_2^2}_{\geq 0} = \|f\|_2^2 - \sum_{i=1}^{n} a_i^2$$

$$\Rightarrow \sum_{i=1}^{n} a_i^2 \leq \|f\|_2^2$$

$$\underset{(n \to \infty)}{\Rightarrow} \sum_{i=1}^{n} a_i^2 \leq \|f\|_2^2$$

The series converges $a_i^2 \to 0$

$\Rightarrow a_i = 0$

< Trigonometric Fourier Series >

$[-\pi, \pi]$, $\left\{ \frac{1}{\sqrt{2\pi}} ; \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}$ $\downarrow$ o.n.s

$$\tilde{a}_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, dx$$

$$\tilde{a}_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$\tilde{b}_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

Fourier Series:

$$\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, dx \cdot \frac{1}{\sqrt{2\pi}}}_{a_0} + \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \cdot \frac{1}{\sqrt{\pi}} \cos(nx) + \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \, \frac{1}{\sqrt{\pi}} \sin(nx) \right)$$

New coefficients:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \leftarrow \text{in front of } \tfrac{1}{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \leftarrow \dots \text{ in front of } \cos(nx)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \leftarrow \dots \text{ in front of } \sin(nx).$$

Example: Trigonometric Fourier Series for $f(x) = x$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x\, dx = 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\, dx = 0 \quad \forall n$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx = -\frac{1}{\pi} \frac{1}{n} \int_{-\pi}^{\pi} x\, d(\cos(nx))\, dx$$

$$= -\frac{1}{\pi n} \left(x \cos(nx) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos(nx)\, dx \right)$$

$$= \frac{1}{-\pi n} \left(\pi \cdot (-1)^n + \pi (-1)^n - 0 \right)$$

$$= \frac{2(-1)^{n+1}}{n}$$

$$: \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx).$$

6-2-2

// Theorem 2.3 (Riemann's Lemma)

let $f \in R[a,b]$, Then $\int_a^b f(x)\cos(\lambda x)\,dx \to 0$

as $\lambda \to \infty$ $\int_a^b f(x)\sin(\lambda x)\,dx \to 0$

Proof: (a) Suppose f is a step-function, that is there is a partition P: $a = t_0 < t_1 < \cdots < t_n = b$
st. $f(x) = a$ for $x \in (t_{i-1}, t_i)$

$$\int_a^b f(x)\cos(\lambda x)\,dx = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \underbrace{f(x)}_{c_i} \cos(\lambda x)\,dx$$

$$= \sum_{i=1}^{n} c_i \underbrace{\int_{t_{i-1}}^{t_i} \cos(\lambda x)\,dx}_{\left.\frac{\sin(\lambda x)}{\lambda}\right|_{t_{i-1}}^{t_i}}$$

$$= \frac{1}{\lambda} \sum_{i=1}^{n} c_i [\sin(\lambda t_i) - \sin(\lambda t_{i-1})]$$

$$\left| \int_a^b f(x)\cos(\lambda x)\,dx \right| \le \underbrace{\frac{1}{\lambda}}_{\to 0} \cdot 2 \max_{1 \le i \le n} |c_i| \cdot n \longrightarrow 0$$

(b) General case: let $f \in R[a,b]$
let $\varepsilon > 0$, Since f is integrable $\exists P$ st. $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$
$a = t_0 < t_1 < \cdots t_n = b$

Define $g(x) = \inf\limits_{x \in (t_{i-1}, t_i)} (x)$, it is a step function

$$\Rightarrow \int_a^b g(x)\cos(\lambda x)\,dx \to 0, \quad \lambda \to \infty$$

$$\exists \lambda_0 \ \forall \lambda \ge \lambda_0, \quad \left| \int_a^b g(x)\cos(\lambda x) \right| < \frac{\varepsilon}{2}$$

$$\left| \int_a^b \underbrace{f(x)}_{g(x)+f(x)-g(x)} \cos(\lambda x)\,dx \right| \le \underbrace{\left| \int_a^b g(x)\cos(\lambda x)\,dx \right|}_{< \frac{\varepsilon}{2}} + \left| \int_a^b \underbrace{(f(x) - g(x))}_{\ge 0} \cos(\lambda x)\,dx \right|$$

⇨

$$< \frac{\varepsilon}{2} + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \underbrace{|(f(x) - g(x))|}_{\geq 0} \, dx$$

$$= \frac{\varepsilon}{2} + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (f(x) - g(x)) \, dx$$

$\underset{x \in (t_{i-1}, t_i)}{\sup f(x)} \leq$ $\cancel{\inf f(x)}$ $\inf f(x)$, $x \in (t_{i-1}, t_i)$

$$\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \left(\sup_{x \in (t_{i-1}, t_i)} f(x) - \inf_{x \in (t_{i-1}, t_i)} f(x) \right)(t_i - t_{i-1})$$

$$= \frac{\varepsilon}{2} + \underbrace{U(f, P) - L(f, P)}_{< \frac{\varepsilon}{2}} < \varepsilon \qquad \square$$

The proof for $\sin(\lambda x)$ is the same.

// $$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)$$

$f(-\pi) = f(\pi)$, extend 2π-periodically on $\mathbb{R}$.

// Thm 2.1: let $f \in R[-\pi, \pi]$; $f(-\pi) = f(\pi)$
Denote by the same symbol f the 2π-periodic ~~extend~~ extension of f to $\mathbb{R}$, Then.

$$S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x-t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t) \, dt$$

where, $$D_n(t) = \begin{cases} \dfrac{\sin(n + \frac{1}{2})t}{\sin \frac{t}{2}} & t \neq 2\pi m \\ 2n+1 & t = 2\pi m \end{cases}$$

Dirichlet kernel

Proof: $S_n(x) = \left[\underbrace{\frac{a_0}{2}}_{\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt} + \sum_{k=1}^{n} \underbrace{a_k}_{\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt} \cos(kx) + \underbrace{b_k}_{\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt} \sin(kx) \right.$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\Big(1 + \sum_{k=1}^{n} 2\underbrace{[\cos(kt)\cos(kx) + 2\sin(kt)\sin(kx)]}_{\cos(k(x-t))}\Big)\,dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\Big(1 + 2\sum_{k=1}^{n}\cos(k(x-t))\Big)\,dt. =$$

$$\sin\frac{\theta}{2}\Big(1 + 2\sum_{k=1}^{n}\cos(k\theta)\Big) = \sin\frac{\theta}{2} + \sum_{k=1}^{n}\Big(\sin\big(\theta(k+\tfrac{1}{2})\big) - \sin\big(\theta(k-\tfrac{1}{2})\big)\Big)$$

$$= \cancel{\sin\frac{\theta}{2}} + \cancel{\sin\frac{3\theta}{2}} - \cancel{\sin\frac{\theta}{2}} + \cancel{\sin\frac{5\theta}{2}} - \cancel{\sin\frac{3\theta}{2}} + \dots \underbrace{\sin\big(\theta(n+\tfrac{1}{2})\big)} - \cancel{\sin\big(\theta(n-\tfrac{1}{2})\big)}$$

$$1 + 2\sum_{k=1}^{n}\cos(k\theta) = \begin{cases} \dfrac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}, & \theta \neq 2\pi m \\ 1+2n, & \theta = 2\pi m \end{cases}$$

$$\dots = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,D_n(x-t)\,dt \qquad \square$$

9-2-20

// Riemann's Lemma: $f \in R[a,b] \Rightarrow \int_a^b f(x)\cos(\lambda x)\,dx \to 0$ as $\lambda \to \infty$

// Dirichlet's Theorem: $f \in R[a,b]$, $f(\pi) = f(-\pi)$

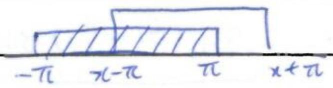
$$\Rightarrow S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_n(x-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_n(t)\,dt$$

where $D_n(t) = \begin{cases} \sin((n+\frac{1}{2})t) & t \neq 2\pi k \\ 2n+1 & t = 2\pi k \end{cases}$

Proof: $S_n(x) = \ldots\ldots = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, D_n(x-t)\,dt = -\frac{1}{2\pi}\int_{x+\pi}^{x-\pi} f(x-\tilde{t})\, D_n(\tilde{t})\,d\tilde{t}$

$\tilde{t} = x - t$

$$= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-t)\, D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_n(t)\,dt \qquad \square$$



// Thm 2.6 let $f \in R[-\pi, \pi]$

$f(-\pi) = f(\pi)$ extend f 2π periodically to R

let $x \in [-\pi, \pi]$, if f is differentiable at x

then $S_n(x) \to f(x)$ as $n \to \infty$

That is, the Trig Fourier Series Converges to $f(x)$ at x

That is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = f(x_0)$

// Thm 2.7: Let $f \in R[-\pi, \pi]$

$f(-\pi) = f(\pi)$ extend f 2π-periodically to R

let $x \in [-\pi, \pi]$

Assume $\exists M > 0,\ \delta > 0$ st $\left|\frac{f(x+t) - f(x)}{t}\right| \leq M \quad \forall t \in (-\delta, \delta)$

Then $S_n(x) \to f(x)$ as $n \to \infty$

Pf: (2.7): $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(1+2\sum_{m=1}^{n}\cos(mt)\right)dt = 1$

$$|S_n(x)-f(x)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt - f(x)\int_{-\pi}^{\pi} D_n(t)\,dt\right|$$

$$= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\left(f(x-t)-f(x)\right)D_n(t)\,dt\right|$$

$$= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \underbrace{\frac{f(x-t)-f(x)}{\sin\frac{t}{2}}}_{g(t)} \sin\left(\left(n+\tfrac{1}{2}\right)t\right)dt\right|$$

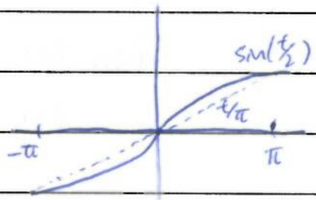
$$g(t) = \begin{cases} \frac{f(x-t)-f(x)}{\sin(t/2)} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

let $\varepsilon > 0$ be small enough s.t. $\varepsilon < 2m\delta$, $g_\varepsilon(t) = \begin{cases} g(t) & t \notin \left[-\frac{\varepsilon}{2m}, \frac{\varepsilon}{2m}\right] \\ 0 & t \in \left[-\frac{\varepsilon}{2m}, \frac{\varepsilon}{2m}\right] \end{cases}$

$$|g(t)| = \left|\frac{f(x-t)-f(x)}{\sin(t/2)}\right| = \underbrace{\left|\frac{f(x-t)-f(x)}{t}\right|}_{\leq m} \cdot \left|\frac{t}{\sin(t/2)}\right|$$

whenever $|t| < \delta$

$$\leq M \cdot \frac{|t|}{|t|} \cdot \pi = M\pi$$



$|\sin\frac{t}{2}| \geq |\frac{t}{\pi}|$

By Riemann's Lemma for g_ε we have
$\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} g_\varepsilon(t)\sin\left(\left(n+\frac{1}{2}\right)t\right)dt\right| < \frac{\varepsilon}{2}$ if $n \geq N$ for some N

Now: $|S_n(x)-f(x)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \underbrace{g(t)}_{g_\varepsilon(t)+g(t)-g_\varepsilon(t)}\sin\left(\left(n+\frac{1}{2}\right)t\right)dt\right|$

$$= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} g_\varepsilon(t)\sin\left(\left(n+\tfrac{1}{2}\right)t\right)dt + \int_{-\pi}^{\pi}\left(g(t)-g_\varepsilon(t)\right)\sin\left(\left(n+\tfrac{1}{2}\right)t\right)dt\right|$$

$$= \underbrace{\frac{1}{2\pi}\left|\int_{-\pi}^{\pi} g_\varepsilon(t)\sin\left(\left(n+\tfrac{1}{2}\right)t\right)dt\right|}_{<\varepsilon/2} + \frac{1}{2\pi}\int_{-\frac{\varepsilon}{2m}}^{\frac{\varepsilon}{2m}}|g(t)-g_\varepsilon(t)|\,dt$$

$$< \frac{\varepsilon}{2} + \frac{1}{2\pi}\int_{-\frac{\varepsilon}{2m}}^{\frac{\varepsilon}{2m}} M\pi\,dt = \frac{\varepsilon}{2} + \frac{1}{2\pi}\cdot M\pi\cdot\frac{\varepsilon}{m} = \varepsilon$$

$\square$

// Pf (2.6): If f is differentiable at x. Then $\lim_{t\to 0} \frac{f(x+t)-f(x)}{t} = f'(x)$

$\forall \varepsilon > 0$ (take $\varepsilon = 1$) $\exists \delta > 0$ st. if $|t| < \delta$

then $\left| \frac{f(x+t)-f(x)}{t} - f'(x) \right| < 1 \Rightarrow \left| \frac{f(x+t)-f(x)}{t} \right| \leq \underbrace{1 + |f'(x)|}_{M} \quad \forall |t| < \delta$

$\underset{\text{Thm 2.7}}{\Rightarrow}$ $S_n(x) \to f(x)$ $\square$

//
$$\frac{1}{\sqrt{2\pi}} a_0 \cdot \sqrt{2\pi} + \sum_{n=1}^{\infty} \sqrt{\pi}\, a_n \cdot \frac{1}{\sqrt{\pi}} \cos(nx) + \sqrt{\pi}\, b_n \cdot \frac{1}{\sqrt{\pi}} \sin(nx)$$

ons.

$$\frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \int_{-\pi}^{\pi} f(x)^2 \, dx \longleftarrow \text{Bessel's Ineq.}$$

// Thm 2.8 (Parseval's Thm)

let ~~[illegible]~~ $f : [-\pi, \pi] \to \mathbb{R}$, $f(\pi) = f(-\pi)$, f is twice differentiable and st. $f'' \in R[-\pi, \pi]$

Then (1) $S_n \to f$ uniformly

(2) $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx$

Pf: (1) $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(x) \, d \sin(nx)$

$$= \frac{1}{n\pi} \Big(\underbrace{f(x)\sin(nx) \Big|_{-\pi}^{\pi}}_{0} - \int_{-\pi}^{\pi} f'(x) \sin(nx) \, dx \Big)$$

$$= -\frac{1}{n^2\pi} \int_{-\pi}^{\pi} f'(x) \, d\,(\cos(nx))\, dx$$

$$= \frac{1}{\pi n^2} \Big(\underbrace{f'(x)\cos(nx) \Big|_{-\pi}^{\pi}}_{0} - \int_{-\pi}^{\pi} f''(x) \cos(nx) \, dx \Big)$$

$$= -\frac{1}{\pi n^2} \int_{-\pi}^{\pi} f''(x) \cos(nx) \, dx$$

$$|a_n| = \frac{1}{\pi n^2} \left| \int_{-\pi}^{\pi} f''(x) \cos(nx) \, dx \right| \leq \frac{1}{\pi n^2} \underbrace{\int_{-\pi}^{\pi} |f''(x)| \, dx}_{m}$$

$|a_n| \leq \frac{m}{\pi n^2}$, Similarly $|b_n| \leq \frac{m}{\pi n^2}$

Now: Since f is differentiable $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$

$$|S_n(x) - f(x)| = \left| \sum_{m=n+1}^{\infty} a_m \cos(mx) + b_m \sin(mx) \right| \leq \sum_{m=n+1}^{\infty} (|a_m| + |b_m|)$$

$$\leq \frac{m}{\pi} \sum_{m=n+1}^{\infty} \frac{1}{n^2} \cdot 2$$

$\| S_n - f \|_{\sup} \leq \frac{2m}{\pi} \sum_{m=n+1}^{\infty} \frac{1}{m^2} \longrightarrow 0$ $\square$

$S_n \to f$

(2) $S_n \to f$ uniformly $\Rightarrow$ $f \cdot S_n \to f^2$ uniformly

$$\uparrow$$

$$\| f \cdot S_n - f^2 \|_{sup} \leq \underbrace{\| f \|_{sup}}_{const} \cdot \underbrace{\| f \cdot S_n \|_{sup}}_{\to 0} \to 0$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x) S_n(x)\, dx \to \int_{-\pi}^{\pi} f^2(x)\, dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\, dx = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) S_n(x)\, dx = \lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\left(\frac{a_0}{2} + \sum_{m=1}^{n} a_m\cos(mx) + b_m\sin(mx)\right)dx$$

$$= \lim_{n\to\infty}\left(\frac{a_0}{2}\cdot a_0 + \sum_{m=1}^{n}(a_m \cdot a_m + b_m \cdot b_m)\right)$$

$$= \cancel{\lim_{n\to\infty}} \frac{a_0^2}{2} + \sum_{m=1}^{\infty}(a_m^2 + b_m^2) \qquad \square$$

20/2/20

<Chapter 3 - Matrics Spaces>

Def: A set X together with a function $d : X \times X \to \mathbb{R}$ is called a metrics space if:

1) $d(x,y) \geq 0 \quad \forall x, y \in X$ and $d(x,y) = 0$ iff $x = y$
2) $d(x,y) = d(y,x) \quad \forall x, y \in X$
3) $d(x,y) \leq d(x,z) + d(z,y), \quad \forall x, y, z \in X$

The function d is called a metric (distance function)

examples: 1) $\mathbb{R}$ is the set of concern, $d(x,y) = |x-y|$

2) $\mathbb{R}^n$, $d(x,y) = \sqrt{\sum_1^n (x_i - y_i)^2}$, $\sqrt{1^2+2^2} = \sqrt{5}$

$d(x,y) = \max\limits_{1 \leq i \leq n} |x_i - y_i|$, $\max\{1,2\} = 2$

$d(x,y) = \sum_{i=1}^n |x_i - y_i|$, $1+2=3$

$d(x,y) = \left(\sum_1^n |x_i - y_i|^q \right)^{\frac{1}{q}}$ with any $q \geq 1$

Discrete metric space:

Any set X and $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

$d(x,y)$ is a metric. If $x = y$, $0 = d(x,y) \leq d(x,z) + d(z,y)$.
since any of $d(x,z)$, $d(z,y)$ can be non-zero.
if $x \neq y$, $1 = d(x,y) \leq \underset{0,1}{d(x,z)} + \underset{\cancel{1,0}\ 0,1}{d(z,y)}$

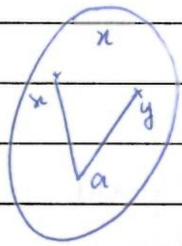
at least one of $d(x,z)$ $d(y,z) = 1$

// Continuous functions :

$C[a,b]$ - continuous function from $[a,b]$ to $\mathbb{R}$

$d(f,g) = \| f-g \|_{sup}$.

// The British Railway Metric :



let $f: X \to \mathbb{R}_{\geq 0}$ where $f(x)$ means the distance from x to a.

$f(x) = 0 \iff x = a$

$\Rightarrow a(x,y) = f(x) + f(y)$

examples : $\mathbb{R}$ $d(x,y) = x^2 + y^2$ ⊗ $d(5,5) = 50 \neq 0$

$d(x,y) = x^2 - y^2$ ⊗ fails all properties eg. $d(5,4) \neq d(4,5)$

$d(x,y) = |x-y|^3$ ⊗ triangle inequality fails, eg

$\underset{8}{d(0,2)} \nleq \underset{1}{d(0,1)} + \underset{1}{d(1,2)}$

On $C[a,b]$ $d(f,g) = \| f-g \|_{sup} + 1$ is not a metric since it is never equal to zero.

// Def^n : A (real) vector space V and a function $\| \cdot \| : V \to \mathbb{R}$ is called a norm space if.

1) $\| x \| \geq 0$ and $\| x \| = 0 \iff x = 0$

2) $\| \lambda x \| = |\lambda| \cdot \| x \|$ $\forall x \in V$, $\forall \lambda \in \mathbb{R}$

3) $\| x+y \| \leq \| x \| + \| y \|$ $\forall x,y \in V$

This function is called a norm

// Lemma (How norm spaces & metric spaces are related)
Let $(V, \|\cdot\|)$ be a normed space, then it is a metric space with $d(x,y) = \|x-y\|$

Pf: Check conditions $d(x,y) \geq 0$ by property 1 of the norm

$\sim$ $d(x,y) = 0 \iff \|x-y\| = 0$
$\iff x-y = 0 \iff x = y$

Symmetry is obvious $d(x,y) = \|x-y\| = \|(-1)(y-x)\|$
$= |-1| \cdot \|y-x\| = \|y-x\| = d(y,x)$

Triangle inequality:
$d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$
$= d(x,z) + d(z,y)$ $\square$

examples:
1) $\mathbb{R}$, $\|x\| = |x|$ is a norm space. In particular, $\mathbb{R}$, $d(x,y) = \|x-y\| = |x-y|$ is a metric space.

2) $\mathbb{R}^n$, $\|x\|_2 = \sqrt{\sum_{i}^{n} x_i^2}$ - 2. norm or euclidian norm
generates metric $d(x,y) = \sqrt{\sum (x_i - y_i)^2}$

generates $d(x,y) = \max |x_i - y_i|$
generates $d(x,y) = \sum_1^n (x_i - y_i)$

$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$\|x\|_1 = \sum_1^n |x_i|$ - 1. norm

$\|x\|_q = \left(\sum |x_i|^q \right)^{\frac{1}{q}}$, q-norm $\forall q \geq 1$

let's prove $\|\cdot\|_\infty$ is a norm.

non-negative ✓

$\|x\|_\infty = 0 \iff \forall |x_i| = 0 \iff x = 0$

$\|\lambda x\|_\infty = \max |\lambda x_i| = |\lambda| \max |x_i| = |\lambda| \cdot \|x\|_\infty$

Triangle inequality: $\|x+y\|_\infty = \max |x_i + y_i| \leq \max(|x_i| + |y_i|)$
$\leq \max |x_i| + \max |y_i| = \|x\|_\infty + \|y\|_\infty$

~~Es~~

Discrete space: cannot be generated by a norm space in general because we don't know whether X is a vector space.

$C[a,b]$: with $\|f\|_{sup}$ generates $d(f,g) = \|f-g\|_{sup}$

British Railway: not a norm space between the set of concern is not a vector

23/2/1[illegible]

// $C[a,b]$ - continuous functions.

$\|f\|_{sup} = \sup_{x \in [a,b]} |f(x)|$ - sup norm

$\|f\|_2 = \sqrt{\int_a^b f(x)^2 \, dx}$ - 2 norm

$\|f\|_1 = \int_a^b |f(x)| \, dx$ - 1-norm

$\|f\|_q = \left(\int_a^b |f(x)|^q \, dx \right)^{q}$ - q-norm $q \geq 1$ (not examable)

~ $\|x\|_2 = \sqrt{\sum_{i=1}^n x^2}$ (Euclidian norm)

to prove this is a norm one uses Cauchy-Schwarz inequality.

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \sqrt{\sum_{i=1}^n x_i^2} \sqrt{\sum_{i=1}^n y_i^2}$$

Similar approach used for $\|f\|_2$.

// Thm 3.1 (Cauchy Schwarz Inequality)

Let $f, g \in R[a,b]$. Then $\left| \int_a^b f(x) g(x) \, dx \right| \leq \left(\int_a^b f(x)^2 dx \right)^{\frac{1}{2}} \left(\int_a^b g(x)^2 dx \right)^{\frac{1}{2}}$

Pf: $\varphi(t) = \| tf - g \|_2^2 \geq 0 \quad \forall t \in \mathbb{R}$

$\varphi(t) = \langle tf - g, tf - g \rangle = \underline{t^2} \|f\|_2^2 - 2\underline{t} \langle f, g \rangle + \|g\|_2^2$

Since $\varphi(t) \geq 0 \quad \forall t$, $D \leq 0$

$D = 4 \langle f, g \rangle^2 - 4 \|f\|_2^2 \|g\|_2^2 \geq 0$

$\langle f, g \rangle^2 \leq \|f\|_2^2 \|g\|_2^2$

$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$ $\square$

// Thm 3.2: $(C[a,b], \|\cdot\|_{sup})$, $(C[a,b], \|\cdot\|_2)$ normed spaces.

Pf ①: $(C[a,b], \|\cdot\|_{sup})$

(a) clear

(b) $\|\lambda \cdot f\|_{sup} = \sup_{x \in [a,b]} |\lambda f(x)| = |\lambda| \cdot \sup_{x\in[a,b]} |f(x)| = |\lambda| . \|f\|_{sup}$.

(c) $\|f+g\|_{sup} = \sup_{x\in[a,b]} |f(x)+g(x)|$

$\leq \sup_{x\in[a,b]} (|f(x)|+|g(x)|)$

$\leq \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |g(x)| \leq \|f\|_{sup} + \|g\|_{sup}$.

②: $(C[a,b], \|\cdot\|_2)$

(a) $\|f\|_2 \geq 0$ clear

$\|f\|_2 = 0 \iff \sqrt{\int_a^b f(x)^2 \, dx} = 0 \iff f(x)^2 = 0 \quad \forall x$

$\iff f(x) = 0 \quad \forall x$

since f is continuous.

(b) $\|\lambda f\|_2 = \sqrt{\int_a^b (\lambda f(x))^2 dx} = |\lambda| \cdot \|f\|_2$

(c) $\|f+g\|_2^2 = \langle f+g, f+g\rangle = \|f\|_2^2 + 2\langle f,g\rangle + \|g\|_2^2$

$\leq \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2$

$= (\|f\|_2 + \|g\|_2)^2$

$\Rightarrow \|f+g\|_2 \leq \|f\|_2 + \|g\|_2$ □

Let (X,d) be a metric space.

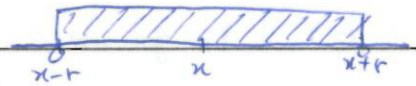
// Def: An open ball : ~~$B^\circ(x,r)$~~ with centre at $x \in X$ of radius $r>0$ is the set $B^\circ(x,r) = \{ y \in X : d(y,x) < r \}$

A closed ball with centre $x \in X$ of radius $r>0$ is the set $B(x,r) = \{ y \in X : d(y,x) \leq r \}$

Example ①: $\mathbb{R}$, $d(x,y) = |x-y|$

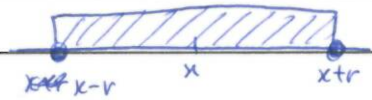
$B^\circ(x,r) \underset{\uparrow}{=} (x-r, x+r)$

$|y-x| < r$



$B(x,r) \underset{\uparrow}{=} [x-r, x+r]$

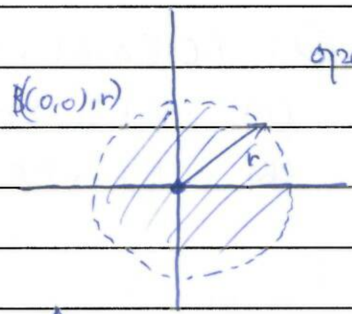
$|y-x| \leq r$



② $\mathbb{R}^2$, $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

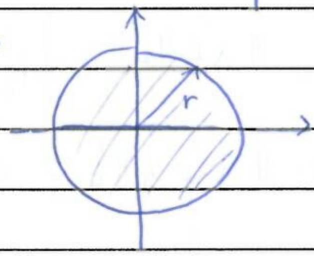
$B^\circ((0,0),r) \longrightarrow$

$\sqrt{y_1^2+y_2^2} < r$



open

$B((0,0),r) \longrightarrow$



closed.

③ $\mathbb{R}^2$ with $\|\cdot\|_\infty$

$d(x,y) = \max\{ |y_1-x_1|, |y_2-x_2| \}$

$B^\circ((0,0),r) = ?$

$d((0,0),(y_1,y_2)) < r$

$\max\{|y_1|,|y_2|\} < r$

$\begin{cases} |y_1| < r \\ |y_2| < r \end{cases}$

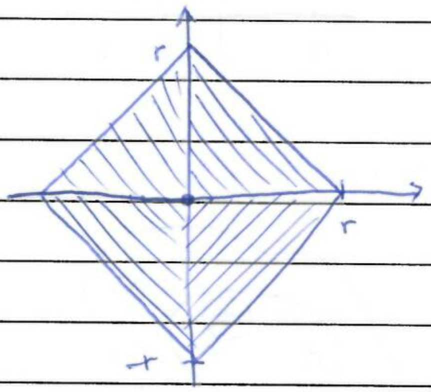
④ $\mathbb{R}^2$ with $\|\cdot\|_1$

$d(x,y) = |y_1 - x_1| + |y_2 - x_2|$

$B^o((0,0), r)$ - ? $d((0,0),(y_1,y_2)) < r$

$|y_1| + |y_2| < r$

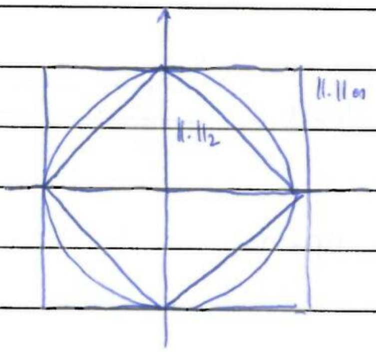
- $y_1, y_2 \geq 0$
  $y_1 + y_2 < r$



- $y_1 \geq 0$, $y_2 \leq 0$
  $y_1 - y_2 < r$

2+3+4

Balls of radius 1 with centre at zero for $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ and $\|\cdot\|_q$, $q \geq 1$



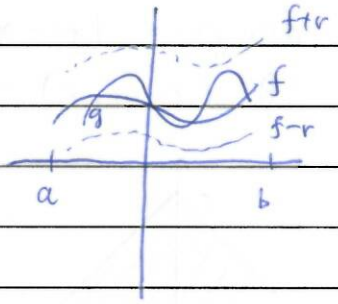
⑤ Discrete space : (a set X with $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

$$B^o(x,r) = \{y \in X : d(x,y) < r\} = \begin{cases} X & \text{if } r > 1 \\ \{x\} & \text{if } r \leq 1 \end{cases}$$

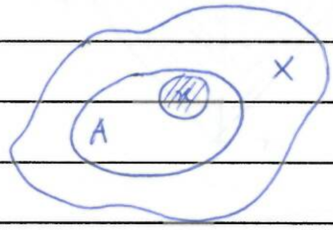
$$B(x,r) = \{y \in X : d(x,y) \leq r\} = \begin{cases} X & \text{if } r \geq 1 \\ \{x\} & \text{if } r < 1 \end{cases}$$

(6) $(C[a,b], \|\cdot\|_{sup})$

$$B^\circ(f,r) = \{ g : \|g-f\|_{sup} < r \}$$



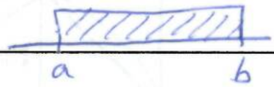
// Def: Let (X,d) be a metric space.
A set $A \subset X$ is open if
$\forall x \in A,\ \exists r > 0$ s.t. $B^\circ(x,r) \subset A$
A set A is closed if $X \setminus A$ is open



It is possible that a set is
- open and closed.
- just open / closed
- neither open nor closed.

Examples: (1) $\mathbb{R}$, $|\cdot|$

(a) $[a,b]$, $[a,\infty)$, $(-\infty,b]$ - closed

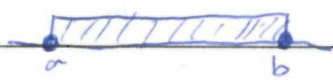


not open

no interval $\subset [a,b]$

(b) (a,b)

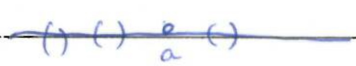
open



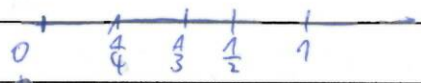
not closed (since $a, b \in \mathbb{R} \setminus (a,b)$ and they cannot be surrounded by a ball $\subset \mathbb{R} \setminus (a,b)$)

(c):

not open



closed

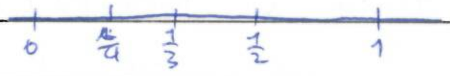
(d)  $\{ \frac{1}{n}, n \in \mathbb{N} \}$

not open

compliment

not closed because cannot find an interval between ~~each~~ interval $(0, \frac{1}{n})$

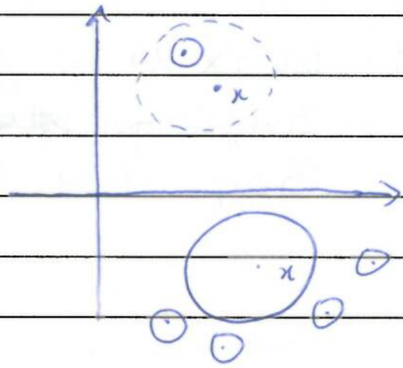
$\{ \frac{1}{n}, n \in \mathbb{N} \} \cup \{0\}$



not open

closed !

(2) $\mathbb{R}^2$, $\| \cdot \|_2$

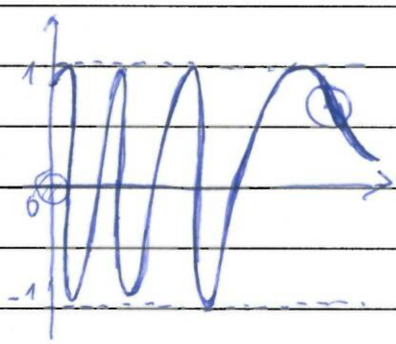


(a) $B^{\circ}(x, r)$ is open

$B(x, r)$ is closed

~~(b)~~

(b). $A = \{ (x, \sin \frac{1}{x}), x > 0 \}$



not open

not closed , look at $x = 0$

$A = \{ (x, \sin \frac{1}{x}), x > 0 \} \cup \{ (0,0) \}$

not open

not closed

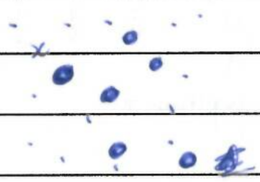
$A = \{ (x, \sin \frac{1}{x}), x > 0 \} \cup \{0\} \times [-1, 1]$

closed !

not open.

③ Discrete space

A



Take any $A \subset X$

Take $x \in A$, $B^0(x, \frac{1}{2}) = \{x\} \subset A$

A is open. Any set in the discrete space is open.

Any set in the discrete space is closed.

Def^n: A set A is open if $\forall x \in A$, $\exists r > 0$ s.t. $B^0(x, r) \subset A$

Thm 3.3: Let (X, d) be a metric space

Every open ball in (X, d) is an open set.

Every closed ball in (X, d) is a closed set.

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Proof a): Consider $B^o(x,r)$

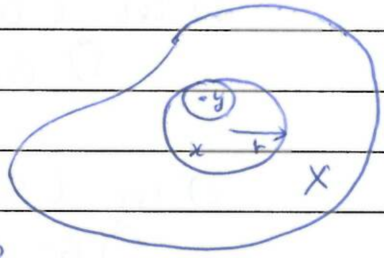
let $y \in B^o(x,r)$ and want to find a ball around y which would fit into $B^o(x,r)$

distance x-y is smaller than r, $d(x,y) < r$

Denote by $p = r - d(x,y) > 0$

Consider $B^o(y,p)$

Want to know $B^o(y,p) \subset B^o(x,r)$



Pick an arbitrary point $z \in B^o(y,p) \Rightarrow d(z,y) < p$

want to show that $z \in B^o(x,r)$

- distance between z and x: $d(z,x) \leq d(z,y) + d(x,y) < p + d(x,y)$

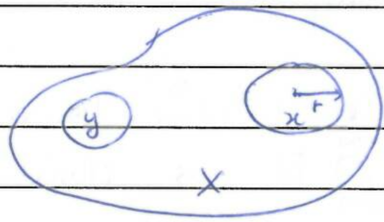
$< r - d(x,y) + d(x,y) = r$

by def^n of p. so $z \in B^o(x,r) \Rightarrow B^o(y,p) \in B^o(x,r)$ so $B^o(x,r)$ is open.

b) Consider $B(x,r)$

Let $y \in X \setminus B(x,r)$

$d(x,y) > r$



Denote by $p = d(x,y) - r > 0$

consider the open ball $B^o(y,p)$ and this is in the compliment of $B^o(y,p) \subset X \setminus B(x,r)$

Pick a point z inside $B^o(y,p)$ and need to show it is not inside $B(x,r)$

$d(z,y) < p$. Want ~~to~~ distance x to $z > r$

$d(x,z) \geq d(x,y) - d(z,y) > d(x,y) - p = r$ by def^n of p

$\therefore z \in X \setminus B(x,r) \Rightarrow B^o(y,p) \subset X \setminus B(x,r)$

so $B(x,r)$ is closed. $\square$

// Thm 3.4: let (X, d) be a metric space

a) ϕ and X (smallest & largest possible subsets - empty set & set X) are open and closed

b) let $(G_i)_{i=1}^{\infty}$ be a collection of open sets. Then $\bigcup_{i=1}^{\infty} G_i$ is open

c) let $(G_i)_{i=1}^{n}$ be a finite collection of open sets. then $\bigcap_{i=1}^{n} G_i$ is open

ex: $G_i = \{-1-1/i, 1+1/i\}$ open in $\mathbb{R}, |\cdot|$
Left end converges to -1, right end converges to 1.
$\bigcap_{i=1}^{\infty} G_i = [-1, 1]$ not open, so c) not true for infinitely many intersections

d) let $(F_i)_{i=1}^{\infty}$ be a collection of closed sets. Then $\bigcap_{i=1}^{\infty} F_i$ is closed.

e) let $(F_i)_{i=1}^{n}$ be a finite collection of closed sets. Then $\bigcap_{i=1}^{\infty} F_i$ is closed.

Not true for infinitely many unions,
eg, take $[-1+\frac{1}{i}, 1-\frac{1}{i}] = F_i$ - closed intervals
Taking union $\bigcup_{i=1}^{\infty} F_i = (-1, 1)$ - not closed.

// Proof a) ϕ : there are values of x in the set which we can surround with balls $\Rightarrow$ open

ϕ closed, since $X \setminus \phi = x$ Take $x \in X$ and $B^\circ(x,r) \subset X$ $\forall r$ ... X is open

X is closed since ϕ is open

X is open since ϕ is closed

b) let $x \in \bigcup_1^\infty G_i \Rightarrow x \in G_i$ for some i

$\therefore$ since G_i open $\exists r > 0$ st $B^\circ(x,r) \subset G_i \subset \bigcup_1^\infty G_i$

c) let $x \in \bigcap_i^n G_i \Rightarrow x \in G_i$ $\forall i$, each G_i is open so $\exists r_i > 0$ st. $B^\circ(x,r) \subset G_i$, which ~~each~~ ball would belong to all the sets? ~ the smallest

So take

$r = \min\{r_1, \dots r_n\} > 0$. Then $B^\circ(x,r) \subset G_i$ $\forall i \Rightarrow B^\circ(x,r) \subset \bigcap_1^n G_i$

d) intersection, closed $\Rightarrow$ compliment is open

$$\underbrace{X \setminus \bigcap_1^\infty F_i = \underbrace{\bigcup_{i=1}^\infty \underbrace{(X \setminus F_i)}_{\text{open}}}_{\text{open}}}$$ $\uparrow$ closed $\leftarrow$ this is open $\Rightarrow \bigcap_1^\infty F_i$ is closed

e) $X \setminus \bigcup_1^\infty F_i = \underbrace{\bigcap_{i=1}^\infty (X \setminus F_i)}_{\text{closed}}$ $\uparrow$ open - open $\Rightarrow \bigcup_1^\infty F_i$ is closed

// Def: let (X,d) be a metric space and let $\{x_n\}_1^\infty$ be a sequence of points in X we say that $x_n \mapsto x \in X$ if $d(x_n, x) \mapsto 0$ distance converges to x

Remark: $x \mapsto x_n \iff d(x_n, x) \mapsto 0 \iff \exists N, \forall \varepsilon > 0$ st. $\forall n \geq N$, $d(x_n, x) < \varepsilon$

$\iff \forall \varepsilon > 0$ $\exists N$ st. $\forall n \geq N$ $x_n \in B^\circ(x, \varepsilon)$

ex: 1) $(\mathbb{R}, |\cdot|)$ convergent sequences are those that converges in Analysis 1.

2) $((0,1), |\cdot|)$, $x_n = \frac{1}{n}$ does not converge in this metric space since 0 is not in $(0,1)$

3) discrete space, $\underbrace{d(x_n, x)}_{0 \text{ or } 1} \mapsto 0 \iff d(x_n, x) = 0$ eventually
$\iff x_n = x$ eventually

So every convergent sequence looks like ...... x x x ...
call such sequences " eventually constant

4) $(C[a,b], \|\cdot\|_{sup})$, $f_n \to f \iff \|f_n - f\|_{sup} \mapsto 0$
$\iff f_n \mapsto f$ unif.

1-3-2...

// $x_n \to x \iff d(x_n, x) \to 0$

Example : (convergent sequences in various metric spaces)

1. $(\mathbb{R}, |\cdot|)$ - standard convergent sequences
2. $((0,1), |\cdot|)$ - $x_n = \frac{1}{n}$ doesn't converge.
3. Discrete space " eventually constant sequences" ...... $x\ x\ x$
4. $(C[a,b], \|\cdot\|_{sup})$ - uniformly convergent sequences.

A is open if $\forall x \in A\ \{\exists r > 0$ st. $B^\circ(x,r) \subset A\}$

is ϕ open - yes

neg.

A is not open if $\exists x \in A$ st. $\{\forall r > 0 \cdots\}$

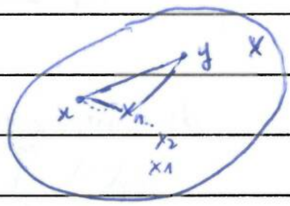
Is ϕ not open? no.

// Thm 3.5 : If $x_n \to x$ and $x_n \to y$ then $x = y$ (ie, limit is unique)

Proof : Suppose $x \neq y$, then $d(x,y) > 0$

$d(x,y) \leq \underbrace{d(x, x_n)}_{\to 0} + \underbrace{d(x_n, y)}_{\to 0} \to 0$

$\Rightarrow d(x,y) = 0$ ↯ □



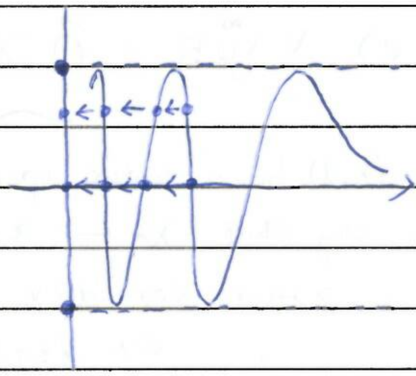
// Thm 3.6 :

Let (X, d) be a metric space

A set $A \subset X$ is ~~closed~~ closed

$\iff$ whenever $x_n \in A\ \forall n$

$x_n \to x$ we have $x \in A$



Proof : Suppose A is closed, but assume there is a sequence x_n st.

RHS $x_n \in A\ \forall n$ and $x_n \to x$ but $x \notin A$

$x \in X \setminus A$, $X \setminus A$ is open $\Rightarrow$

$\exists r > 0$ st. $B^\circ(x,r) \subset X \setminus A$

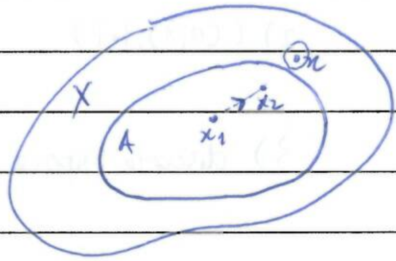
But $x_n \to x \Rightarrow d(x_n, x) \to 0$

$\Rightarrow \exists N\ \forall n \geq N \quad d(x_n, x) < r$

$\Rightarrow x_n \in B^\circ(x,r)$

$\Rightarrow x_n \in X \setminus A$

But $x_n \in A$ ↯



LHS : Suppose the RHS is true. But assume that A is not closed $\Rightarrow X \setminus A$ is not open

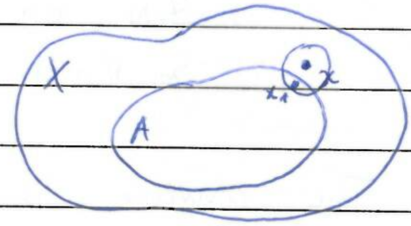
$\Rightarrow \exists x \in X \setminus A$ st. $\forall r > 0$

$B^{\circ}(x,r) \cap A \neq \phi$

take $r = \frac{1}{n}$, $n \in \mathbb{N}$

$\Rightarrow B^{\circ}(x, \frac{1}{n}) \cap A \neq \phi$

Pick $x_n \in B^{\circ}(x, \frac{1}{n}) \cap A$



- $x_n \in A \quad \forall n$
- $d(x_n, x) < \frac{1}{n} \to 0 \Rightarrow x_n \to x$

(distance between x_n and x)

- But $x \notin A$!

contradiction to the RHS

$\Rightarrow A$ is closed. $\square$

// Def: A sequence $\{x_n\}$ in a metric space (X,d) is a Cauchy-sequence if $\forall \varepsilon > 0 \ \exists N \ \forall n,m \geq N$, $d(x_n, x_m) < \varepsilon$

$|x_n - x_m| < \varepsilon$

Examples: ① $(\mathbb{R}, |\cdot|)$ - standard Cauchy sequences.

② $([0,1], |\cdot|)$ - $x_n = \frac{1}{n}$, check $|\frac{1}{n} - \frac{1}{m}| < \varepsilon$

$\forall n,m \geq N \Rightarrow |\frac{1}{n} - \frac{1}{m}| < \frac{1}{n} + \frac{1}{m} < \frac{2}{N} < \varepsilon$

- Cauchy sequence. Choose N st. $\frac{1}{N} < \varepsilon$

③ : Discrete space : $\{x_n\}$ is Cauchy if $d(x_n, x_m) < \varepsilon$

$\Rightarrow d(x_n, x_m) = 0$ eventually (either 0 or 1) (take $\varepsilon = \frac{1}{2}\varepsilon$)

$\Rightarrow x_n = x_m$ eventually

$\Rightarrow$ Cauchy sequence $\to$ constant sequences.

④ $(C[a,b], \|\cdot\|_{sup})$

$\forall \varepsilon > 0 \ \exists N \ \forall n,m \geq N \quad \|f_n - f_m\|_{sup} < \varepsilon$

so Cauchy sequence in this space are those that we called "uniform Cauchy Sequences"

// Lemma : Every convergent sequence is a Cauchy sequence

Proof : ~~$\forall \varepsilon>0$~~ Suppose $x_n \to x$, let $\varepsilon > 0$

$\exists N \ \forall n \geq N \quad d(x_n, x) < \frac{\varepsilon}{2}$

So $\forall n, m \geq N \quad d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

The converse is not true !! see Ex②

// Def : let (X, d) be a metric space.

It is called a complete metric space if every Cauchy sequence in this space converges.

A complete normed space is called Banach space

// Example ① : $(\mathbb{R}, |\cdot|)$ complete space (Banach space)

② : $((0,1), |\cdot|)$ not complete as $\frac{1}{n}$ is Cauchy but not converge.

③ : Discrete space is complete but not a Banach sp. since it is not a normed space.

④ : $(C[a,b], \|\cdot\|_{\sup})$ - complete by CPUC
- Banach space.

⑤ $(\mathbb{Q}, |\cdot|)$

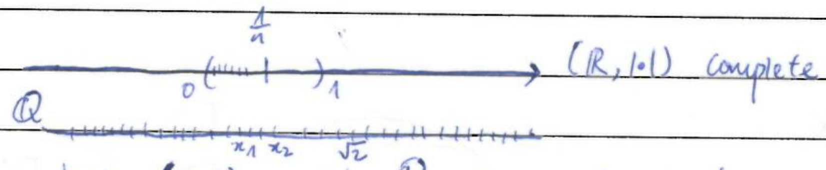
eg $x_n \to \sqrt{2}$ (x_n rational) doesn't converge in $(\mathbb{Q}, |\cdot|)$

it converges in $(\mathbb{R}, |\cdot|)$

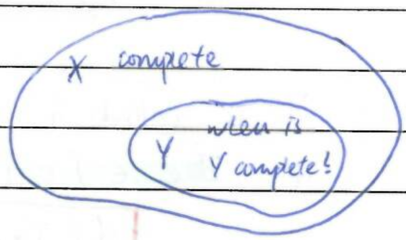
not complete ! so it is Cauchy in $(\mathbb{R}, |\cdot|)$

$\Rightarrow$ it's Cauchy in $(\mathbb{Q}, |\cdot|)$

⑥ $\mathbb{R}^n$, $\left.\begin{array}{l} \|\cdot\|_1 \\ \|\cdot\|_2 \\ \|\cdot\|_\infty \\ \|\cdot\|_q \end{array}\right\}$ all are complete and so Banach.



but $(0,1)$ and $\mathbb{Q}$ are not complete.

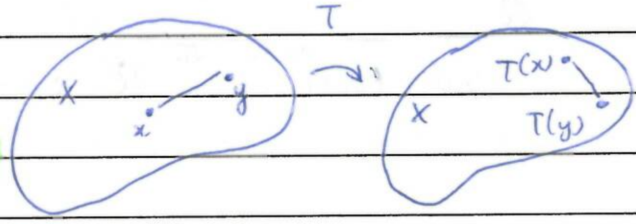


// Thm 3.7: Let (X,d) be a complete metric space let $Y \subset X$. Then Y (with the same metric space as X) is complete $\Leftrightarrow$ Y is closed.

Proof: Suppose Y is complete (RHS)
let $y_n \in Y$ $\forall n$ and $y_n \to y \in X$
Then (y_n) is Cauchy in X.
$\Rightarrow (y_n)$ is Cauchy in Y
$\Rightarrow y_n$ converges in $Y \Rightarrow y \in Y$
By Thm 3.6, Y is closed.

(LHS): Suppose Y is closed then let (y_n) be a Cauchy sequence in (Y,d). Then (y_n) is a Cauchy sequence in (X,d).
Therefore $y_n \to y \in X$ (since X is complete)
By Thm 3.6 $y \in Y$
$\Rightarrow$ y_n converges in $Y \Rightarrow (Y,d)$ is complete $\square$

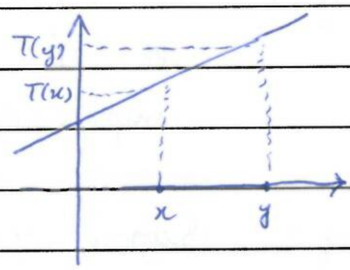
// Def: Let (X, d) be a metric space and $T: X \to X$



T is called a contraction mapping if $\exists\, c \in [0,1]$ s.t.

$$\boxed{d(Tx, Ty) \le c\, d(x,y)}$$

for all $x, y \in X$.



Example: ① $(\mathbb{R}, |\cdot|)$, $T(x) = \frac{x}{3} + 2$

$|T(x) - T(y)| = \left|\frac{x}{3} + 2 - \frac{y}{3} - 2\right| = \underbrace{\tfrac{1}{3}}_{= c}|x-y|$

contraction mapping with $c = \frac{1}{3}$

②: $(\mathbb{R}, |\cdot|)$

$T(x) = \sin\left(\frac{x}{2}\right)$, ~~Intermediate~~ Mean Value Thm

$|T(x) - T(y)| = \left|\sin\left(\frac{x}{2}\right) - \sin\left(\frac{y}{2}\right)\right| = |T'(\xi)|\,|x-y| \le \frac{1}{2}|x-y|$

$\Rightarrow \frac{1}{2}|\cos(\xi)| \le \frac{1}{2}$

contraction mapping with $c = \frac{1}{2}$

In general, one can compute $\|T'\|_{sup}$ if $\|T'\|_{sup} < 1$ then T is a contraction mapping.

③: $([1, \infty), |\cdot|)$

$T(x) = x + \frac{1}{x}$, $T'(x) = 1 - \frac{1}{x^2} < 1 \quad \forall x$

$\|T'\|_{sup} = 1 \leftarrow$ the MVT cannot be used.

Maybe T is still a contraction mapping?

Suppose it is! $\Rightarrow \exists\, c \in (0,1)$ s.t.

$|T(x) - T(y)| \le c\,|x-y|$

$\left|\dfrac{x + \frac{1}{x} - y - \frac{1}{y}}{x-y}\right| \le c \quad \forall x, y$

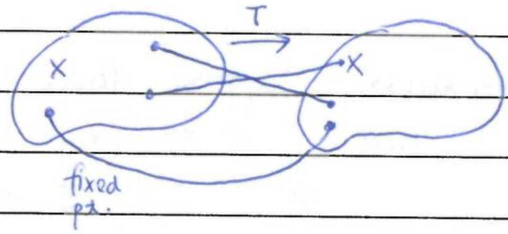
Take $y = 2x$ and let $x \to \infty$

$\lim_{x\to\infty} \left|\dfrac{x + \frac{1}{x} - 2x - \frac{1}{2x}}{x - 2x}\right| \le c$

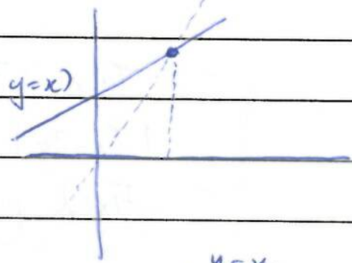
$\Leftrightarrow \lim_{x\to\infty} \left|\dfrac{x - \frac{1}{2x}}{x}\right| \le c \rightsquigarrow 1 \le c$ ↯

// **Def** : Let (X,d) be a metric space and $T: X \to X$

$x \in X$ is a fixed point of T if $T(x) = x$



// **Thm 3.8** : Let (X,d) be a non-empty complete metric space and let $T: X \to X$ be a contraction mapping. Then T has a unique fixed point.



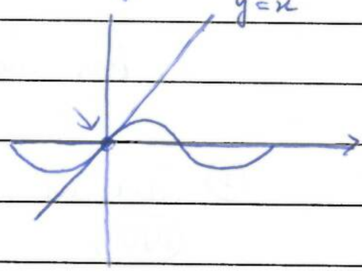
Examples ① : $T(x) = \frac{x}{3} + 2$, (look at intersection with $y=x$)

$\frac{x}{3} + 2 = x \Rightarrow x = 3$

~~②: T=sin~~

② : $T(x) = \sin\left(\frac{x}{2}\right)$

$x=0$ is the unique fixed point.



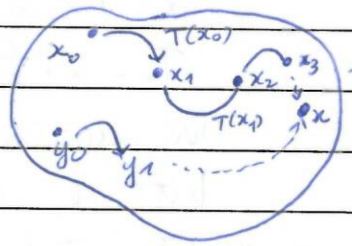
Proof: let $x_0 \in X$ be any point

Define $x_n = Tx_{n-1}$ $\forall n \geq 1$

To prove $\{x_n\}$ is Cauchy

$d(x_n, x_{n-1}) = d(Tx_{n-1}, Tx_{n-2}) \leq c\, d(x_{n-1}, x_{n-2})$

$\leq c^{n-1} d(x_1, x_0)$



$n > m$

$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \dots + d(x_{m+1}, x_m)$

$= \sum_{i=m}^{n-1} d(x_{i+1}, x_i) \leq \sum_{i=m}^{n-1} c^i d(x_1, x_0) \leq d(x_1, x_0) \sum_{i=m}^{\infty} c^i$

$= d(x_1, x_0) \frac{c^m}{1-c} \longrightarrow 0$ as $m \to \infty$

$\Rightarrow \{x_n\}$ is a Cauchy sequence

$\Rightarrow \{x_n\}$ converges to some $x \in X$ since (X,d) is complete.

Lets prove x is a fixed point, ie $Tx = x$

$x_n \to x$

$x_{n+1} \to x \Rightarrow \cancel{d(x_{n+1}, x) \to 0}$

$d(x_{n+1}, Tx) = d(Tx_n, Tx) \leq c\, d(x_n, x) \to 0$ ⇨

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$\Rightarrow \quad x_{n+1} \to Tx$

so $\quad Tx = x$ $\boxtimes$

Uniqueness: Suppose there are two fixed points $x \neq y$

Then, $\quad 0 \neq d(x,y) = d(Tx,Ty) \leq c \cdot d(x,y)$

$\Rightarrow \quad 1 \leq c$, Contradiction since $c < 1$ $\square$

// Examples: ① Is the completeness of X important? Yes

Take $X = (0,\infty)$ with $|\cdot|$ - incomplete

$x_n = \frac{1}{n}$ Cauchy but doesn't converge

Take $Tx = \frac{x}{2}$ - contraction mapping

$|Tx - Ty| = \frac{1}{2}|x-y|$

Fixed point? $\quad Tx = x$

$\frac{x}{2} = x$, $x = 0 \notin (0,\infty)$

has no solution in $(0,\infty)$

② Can we replace $d(Tx,Ty) \leq c\,d(x,y)$ by $d(Tx,Ty) < d(x,y)$

NO!

$X = [1,\infty)$ - complete space.

$Tx = x + \frac{1}{x}$

We discussed that it's not a contraction mapping

but it satisfies $d(Tx,Ty) < d(x,y)$:

* $|Tx - Ty| = |T'(\xi)|\,|x-y| < |x-y|$

$\sim T'(x) = 1 - \frac{1}{x^2} \Rightarrow |T'(\xi)| < 1$

Fixed point! $\quad Tx = x$

$x + \frac{1}{x} = x$ no solution

// Application of CMT to diff. equation

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

for example $\begin{cases} y' = xy \\ y(0) = 1 \end{cases}$

$$\frac{dy}{y} = x\,dx$$

$$\ln(y) = \frac{x^2}{2} + \tilde{c}$$

$$\boxed{y = ce^{\frac{x^2}{2}}} \Rightarrow y(x) = e^{\frac{x^2}{2}}$$

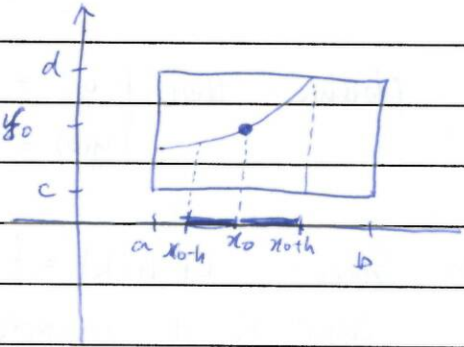
// Thm 3.9: (Picard Theorem)

let $P = [a,b] \times [c,d]$

let $f : P \to \mathbb{R}$ be such that

- f is continuous on P
- $\frac{\partial f}{\partial y}$ is continuous on P

let $(x_0, y_0) \in (a,b) \times (c,d)$



Then $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ has a unique solution on $[x_0 - h_0, x_0 + h_0]$ for some $h_0 > 0$

Idea: Integrate $y' = f(x,y)$ over $[x_0, x]$

$$y(x) - \underbrace{y(x_0)}_{y_0} = \int_{x_0}^{x} f(x, y(x))\,dx$$

$$y(x) = \underbrace{y_0 + \int_{x_0}^{x} f(x, y(x))\,dx}$$ ← $y(x)$ has to solve this !

$$T : \varphi \longmapsto y_0 + \int_{x_0}^{x} f(x, y(x))\,dx$$

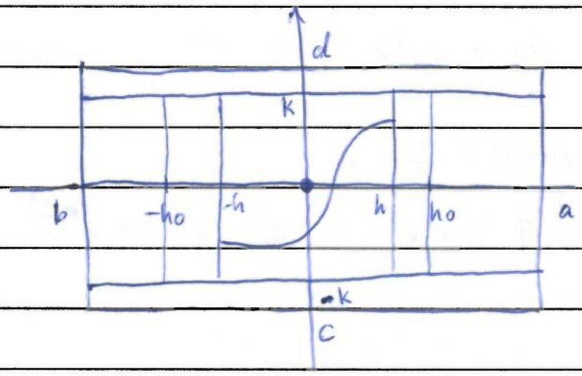
Prove T is a contraction mapping, then it has a unique fixed point which is our solution.

(?) what is our metric space.

→ that is if there is another solution defined on a smaller interval, it must coincide with our solution on that interval.

Proof: (Picard theorem): NOT EXAMABLE

① Without loss of generality $x_0 = 0$, $y_0 = 0$



Denote $M = \sup\limits_{(x,y)\in P} f(x,y) < \infty$

$M' = \sup\limits_{(x,y)\in P} \dfrac{\partial f(x,y)}{\partial y} < \infty$

Choose $k < |c|, |d|$

$h_0 = \min\left\{|a|, |b|, \dfrac{k}{2M}, \dfrac{1}{2M'}\right\}$

Take any $h \in [0, h_0]$

Observe that $\begin{cases} y' = f(x,y) \\ y(0) = 0 \end{cases} \Leftrightarrow \begin{cases} y(x) = \int_0^x f(t, y(t))\,dt \\ y \text{ is continuous} \end{cases}$

②: Space: $C(h,k) = \{\psi : [-h,h] \to \mathbb{R}, \text{ continuous}, \|\psi\|_{\sup} \leq k\}$ / with $\|\psi\| = \sup\limits_{x\in[-h,h]} |\psi(x)|$

Why is it complete?

$C(h,k) \subset \underbrace{C[-h,h]}_{\text{complete}}$

It suffices to prove that $C(h,k)$ is closed

Let $\{f_n\}$ be a sequence from $C(h,k)$ s.t. $f_n \xrightarrow[\text{uniformly}]{} f$

Since $|f_n(x)| \leq k \quad \forall x$

$\downarrow$

$|f(x)| \leq k \Rightarrow \|f\| \leq k$

$\Rightarrow f \in C(h,k)$

$\Rightarrow C(h,k)$ is complete.

③: Mapping: $T : C(h,k) \to C(h,k)$

$\psi \longmapsto \int_0^x f(t, \psi(t))\,dt$ } this function is indeed in $C(h,k)$ because its continuous and

$$\sup_{x\in[-h,h]} \Big| \int_0^x \underbrace{f(t,\psi(t))}_{\leq M}\,dt \Big| \leq M \cdot h \leq M h_0 \leq M \cdot \frac{k}{2M} = \frac{k}{2} \leq k$$

④ T is a contraction mapping

let $\psi_1, \psi_2 \in C(h,k)$

$$\| T\psi_1 - T\psi_2 \|_{sup} = \sup_{x\in[-h,h]} \left| \int_0^x f(t,\psi_1(t))\,dt - \int_0^x f(t,\psi_2(t))\,dt \right|$$

$$= \sup_{x\in[-h,h]} \left| \int_0^x \underbrace{\left(f(t,\psi_1(t)) - f(t,\psi_2(t)) \right)}_{\frac{\partial f}{\partial y}(\xi(t))\cdot(\psi_1(t)-\psi_2(t))} dt \right|$$

$$\underset{MVT.}{\leq} \sup_{x\in[-h,h]} \left| \int_0^x \underbrace{\frac{\partial f}{\partial y}(\xi(t))}_{\leq M'} \underbrace{(\psi_1(t)-\psi_2(t))}_{\leq \|\psi_1(t)-\psi_2(t)\|_{sup}} dt \right|$$

$$\leq h \cdot M' \|\psi_1 - \psi_2\|_{sup} \leq \underbrace{h_0 M'}_{\leq \frac{1}{2M'}M'} \|\psi_1 - \psi_2\|_{sup} \leq \frac{1}{2} \|\psi_1 - \psi_2\|_{sup}$$

⑤ By CMT: The mapping T on $C(h,k)$ has a unique fixed point, that is, for each $0 < h \leq h_0$ our diff. equation has a unique solution φ_h on $[-h,h]$

In particular, our diff. equation has a solution φ_{h_0} on $[-h_0,h_0]$ (bounded by k)

⑥ Uniqueness: Suppose there is another solution φ

$\Rightarrow \|\varphi\| > k$

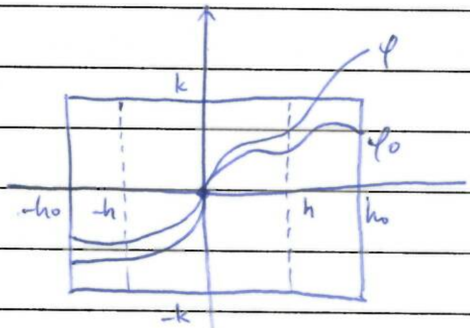
$h = \inf\{ |x| : \varphi(x) = k \}$

Restrict φ on $[-h,h]$

$\|\varphi\| = k$ on $[-h,h]$ & φ is a solution

$\Rightarrow \varphi = T\varphi$ on $[-h,h]$

$\Rightarrow \underbrace{\|\varphi\|}_{k} = \underbrace{\|T\varphi\|}_{\leq \frac{k}{2}}$ on $[-h,h]$ ↯ (step ③)



$\Rightarrow \therefore$ uniqueness

// An application of Picard's Theorem

$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases} \iff \underbrace{y(t) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt}$$

$$T: \varphi \to y_0 + \int_{x_0}^{x} f(t, y(t))\,dt.$$

solution y is the fixed point of T.

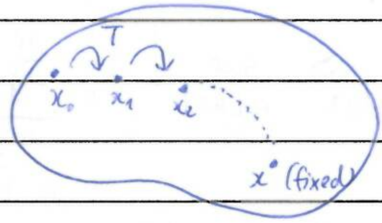
Picard iterates:

$\varphi_0(x) = y_0 \quad \forall x$

$\varphi_1(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_0(t))\,dt$

$\varphi_2(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_1(t))\,dt$

and so on for all n



Then according to the proof of CMT. $\varphi_n(x) \to \underbrace{y(x)}_{\text{solution}}$ pointwise

Example: $\begin{cases} y' = xy \\ y(0) = 1 \end{cases} \qquad y(x) = e^{\frac{x^2}{2}} = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \dots$

$\varphi_0(x) = 1$

$\varphi_1(x) = 1 + \int_0^x t \cdot \underbrace{\varphi_0(t)}_{1}\,dt = 1 + \frac{x^2}{2}$

$\varphi_2(x) = 1 + \int_0^x t \cdot \underbrace{\left(1 + \frac{t^2}{2}\right)}_{\varphi_1(t)}\,dt = 1 + \frac{x^2}{2} + \frac{x^4}{8}$

Def^n: Let (X, d_x) and (Y, d_y) be two metric spaces and let

$$f : X \to Y$$

Let $a \in X$, we say that $\lim_{x \to a} f(x) = b \in Y$

If $\forall \varepsilon > 0 \ \exists \delta > 0$ if $0 < d_x(x, a) < \delta$ then $d_y(f(x), b) < \varepsilon$

We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a)$$

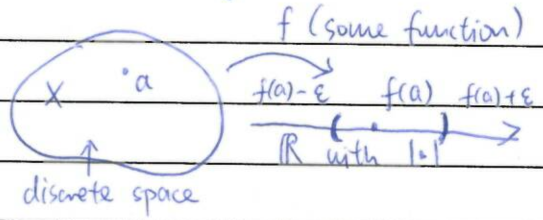
We say that f is continuous if it is continuous at each point $a \in X$

Remark: f is continuous at $a \in X$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ st.
if $d_x(x, a) < \delta$ then $d_y(f(x), f(a)) < \varepsilon$

Equivalently, $\forall \varepsilon > 0, \exists \delta > 0$ st. $f(B^\circ(a, \delta)) \subset B^\circ(f(a), \varepsilon)$

Examples: ① If $f : \mathbb{R} \to \mathbb{R}$ with $|\cdot|$ then we have the standard definition of continuity.

②



Every function from the discrete space to $(\mathbb{R}, |\cdot|)$ is continuous.

Let $\varepsilon > 0$, choose $\delta = \frac{1}{2}$

Then $B^\circ(a, \frac{1}{2}) = \{a\}$ and $f(\{a\}) \subset (f(a) - \varepsilon, f(a) + \varepsilon)$

③ $(C[0,1], \|\cdot\|_2)$ $\quad (C[0,1], \|\cdot\|_1)$

Identity mapping $F : C[0,1] \to C[0,1]$

$$f \longmapsto f$$

Is F continuous?

Let $f \in C[0,1]$, let $\varepsilon > 0$ choose $\delta \overset{=\varepsilon}{\ldots\ldots}$ If $\|g - f\|_2 < \delta$

$$\|F(g) - F(f)\|_1 = \|g - f\|_1 = \int_0^1 1 \, |f(x) - g(x)| \, dx$$

$$\leq \sqrt{\int_0^1 1^2 dx} \sqrt{\int_0^1 |f(x) - g(x)|^2 dx}$$

Cauchy Schwarz Ineq. $\quad = \|g - f\|_2 < \delta = \varepsilon$

④: $(C[0,1], \|\cdot\|_1)$ $\quad (C[0,1], \|\cdot\|_2)$

Identity mapping $F(f) = f$, ~~Disconti~~ Discontinuous!

// Thm 3.10: f is continuous at $a \iff$ for any sequence $x_n \to a$ one has $f(x_n) \to f(a)$

Proof: Suppose f is continuous at a. Suppose $x_n \to a$. let $\varepsilon > 0$. Then by continuity $\exists\, \delta > 0$ st. if $d_X(x, a) < \delta$

$d_Y(f(x), f(a)) < \varepsilon$

Since $x_n \to a$ we have $d(x_n, a) \longrightarrow 0$

$\Rightarrow \exists N$ st. $\forall n \geq N \quad d_X(x_n, a) < \delta$

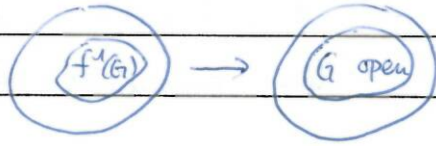
So $d_Y(f(x_n), f(a)) < \varepsilon$

$\Rightarrow \quad f(x_n) \to f(a)$ $\square$

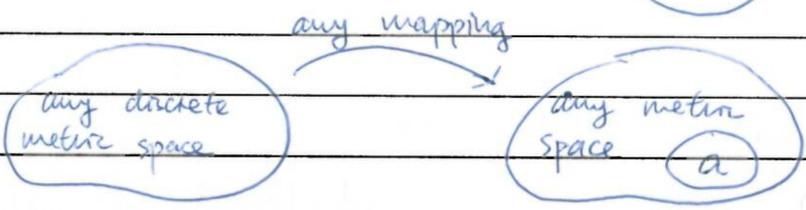
// $f: X \to Y$ is continuous at $a \in X$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ st. $f(B^{\circ}(a,\delta)) \subset$

or equivalently if for any $x_n \to a$ we have $f(x_n) \to f(a)$

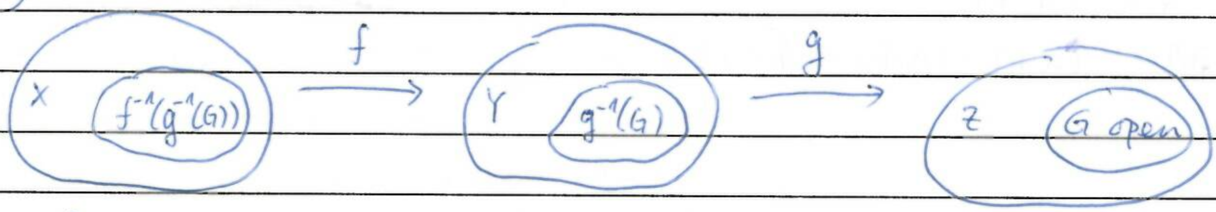
Thm: $f: X \to Y$ is continuous (everywhere) $\iff$ for any open set $G \subset Y$ the ~~percentage~~ preimage $f^{-1}(G)$ is open in X



Examples ①:



②



Suppose f, g are continuous

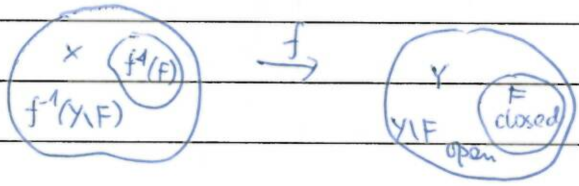
Then $g \circ f$ is continuous: take G (open set in Z)

$\Rightarrow$ $g^{-1}(G)$ is open in Y since g is continuous.

$\Rightarrow$ $\underbrace{f^{-1}(g^{-1}(G))}_{(g \circ f)^{-1}(G)}$ is open in X since f is continuous

$(g \circ f)^{-1}(G)$ open

③: It is also true that f is continuous $\iff$ the preimage of every closed set is closed.



F is closed $\iff$ & $Y \setminus F$ is open

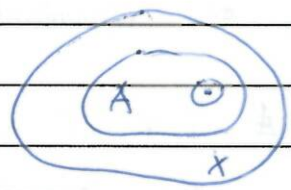
$f^{-1}(F)$ is closed $\iff$ $f^{-1}(Y \setminus F)$ open.

11.11

// Def^n: Let (X, d) be a metric space and let $A \subset X$

The interior A° of A is

$A^\circ = \{ x \in A \quad \exists r > 0 \text{ st. } B^\circ(x, r) \subset A \}$



The closure $\bar{A}$ of A is

$\bar{A} = \{ x \in X$, there is a sequence $x_n \in A$ st. $x_n \to x \}$

The boundary ∂A of A is $\partial A = \bar{A} \setminus A^\circ$

Examples ① $(\mathbb{R}, |\cdot|)$

$A^\circ = (a, b)$ $\qquad A = [a, b]$

$\bar{A} = [a, b]$

$\partial A = [a, b] - (a, b) = \{a, b\}$

②: $(\mathbb{R}, |\cdot|)$ $\qquad A = \{ \frac{1}{n}, n \in \mathbb{N} \}$

$A^\circ = \phi$

$\bar{A} = \{ \frac{1}{n}, n \in \mathbb{N} \} \cup \{0\}$

$\partial A = \{ \frac{1}{n}, n \in \mathbb{N} \} \cup \{0\}$

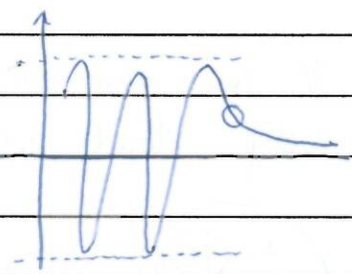
③ $\mathbb{R}^2$, Euclidean distance

$A = \{ (x, \sin \frac{1}{x}) : x > 0 \}$

$A^\circ = \phi$

$\bar{A} = A \cup \{0\} \times [-1, 1]$

$\partial A = A \cup \{0\} \times [-1, 1]$



R

// Thm 3.12 : (1) $A^\circ \subset A$

(2) A° is open set

(3) if A is open than $A^\circ = A$

(4) $A \subset \bar{A}$

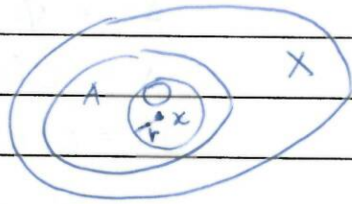
(5) $\bar{A}$ is a closed set

(6) if A is closed then $\bar{A} = A$

Proof : (1) obvious

(2) let $x \in A^\circ$

$\Rightarrow \exists r > 0 \quad B^\circ(x,r) \subset A$



Take any $y \in B^\circ(x,r)$

Since $B^\circ(x,r)$ is open

$\exists \delta > 0$ st. $B^\circ(y,\delta) \subset B^\circ(x,r) \subset A$

$\Rightarrow y \in A^\circ$

$\Rightarrow B^\circ(x,r) \in A^\circ \Rightarrow A^\circ$ is open

(3) obvious

(4) If $x \in A$ then $x, x, x, \dots \to x \Rightarrow x \in \bar{A}$

(5) let $y_n \in \bar{A}$ such that $y_n \to y \in X$

It suffices to prove that $y \in \bar{A}$

Since each $y_n \in \bar{A}$

there are sequences $x_m^{(n)} \in A \xrightarrow[m \to \infty]{} y_n$

$\forall n$ choose $d(x^{n}_{m(n)}, y_n) < \frac{1}{n}$

Now $d(x^{(n)}_{m(n)}, y) \leq \underbrace{d(x^{(n)}_{m(n)}, y_n)}_{< \frac{1}{n} \to 0} + \underbrace{d(y_n, y)}_{\to 0} \to 0$

$x^{(n)}_{m(n)} \to y \Rightarrow y \in \bar{A}$ □

(6) $y \in \bar{A} \Rightarrow \underset{\in A}{x_n} \to y$

Since A is closed $y \in A$

$\Rightarrow \bar{A} \subset A$

$\Rightarrow \bar{A} = A$

Examples ①: Discrete space

A - any set

$A^\circ = A$ (since A is open)

$\overline{A} = A$ (since A is closed)

$\partial A = \phi$

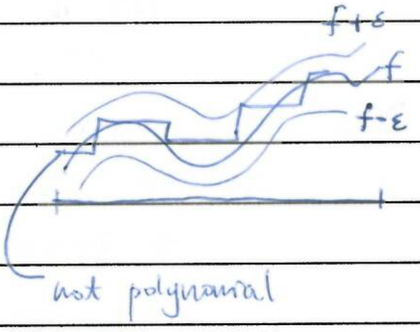
②: $(C[a,b], \|\cdot\|_{sup})$

A = set of all polynomials

$A^\circ = \phi$

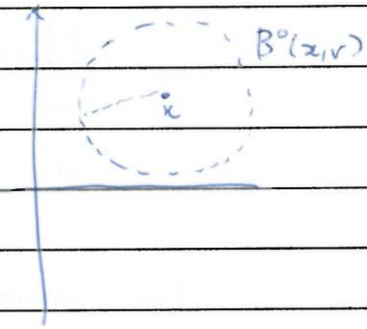
$\overline{A} = C[a,b]$ (by Weierstrass Approx. Theorem)

$\partial A = C[a,b]$



③: In $\mathbb{R}^2$ with Euclidean distance

$B^\circ(x,r) = B(x,r)$



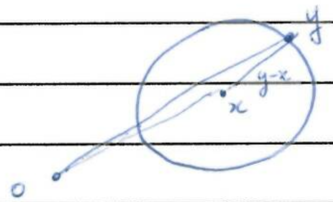
Is this always true (in any metric space)?

NO (X,d) - discrete space

$B^\circ(x,1) = \{x\}$

$\overline{B^\circ(x,1)} = \{x\}$ but $B(x,1) = X$

But its still true in any normed space.



look at $B^\circ(x,r)$

Take y st. $\|y-x\| = r$

Take $y_n = x + (y-x)(1-\frac{1}{n})$

$y_n \in B^\circ(x,r)$: $\|y_n - x\| = \|x + (y-x)(1-\frac{1}{n}) - x\|$

$= (1-\frac{1}{n})\underbrace{\|y-x\|}_{r} < r$

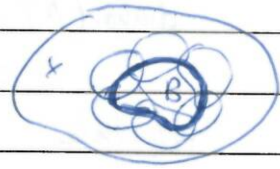
$y_n \to y$: $\|y_n - y\| = \|x + (y-x)(1-\frac{1}{n}) - y\|$

$= \|\frac{1}{n}(x-y)\| = \frac{1}{n}\underbrace{\|x-y\|}_{r} = \frac{r}{n} \to 0$

$\Rightarrow y \in \overline{B^\circ(x,r)}$

$\Rightarrow \overline{B^\circ(x,r)} = B(x,r)$

// Def: Let (X,d) be a metric space,
A collection of open sets $\{G_\alpha\}_{\alpha \in A}$ is a cover for a set B if $B \subset \bigcup_{\alpha\in A} G_\alpha$

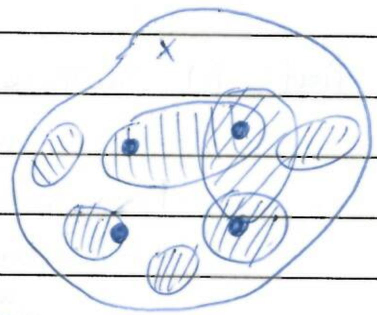


A subcover of a cover $\{G_\alpha\}_{\alpha\in A}$ is a subcollection which itself is a cover for B.

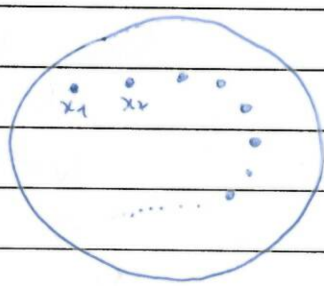
A set B is compact if any cover of B has a finite subcover.

Examples ①: $(\mathbb{R}, |\cdot|)$ and $B = [a,b]$
$[a,b]$ is compact by Heine-Borel Thm.

②: (X,d) - discrete metric space
$B \subset X$ - compact ?



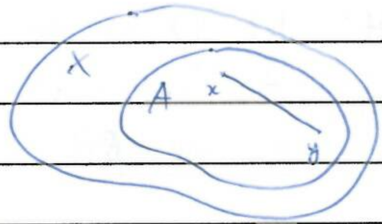
- Suppose B is finite
  Let $\{G_\alpha\}$ be an arbitrary cover
  Since only need to cover finitely many points, there is a finite subcover
  $\Rightarrow$ B is compact.



- Suppose B is infinite
  $\underset{\text{open}}{\{x_1\}}, \underset{\text{open}}{\{x_2\}}, \ldots\ldots$ - a cover for B
  In general, take $\{x\}$ for each $x \in B$
  This is a cover, but it has no finite subcover
  $\Rightarrow$ B is not compact.

Def: let (X, d) be a metric space and $A \subset X$

$$\operatorname{diam}(A) = \sup_{x,y \in A} d(x,y)$$



Is the diameter of A

We say that A is bounded if $\operatorname{diam}(A) < \infty$

Example: $\text{diameter}(A) = \begin{cases} 0 & \text{if } A \text{ consists of } 1 \text{ element} \\ 1 & \text{otherwise.} \end{cases}$

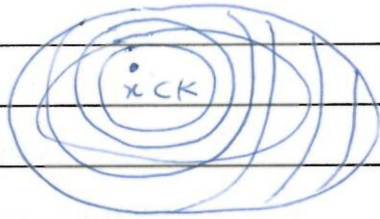
a set in the discrete space

Any set in the discrete space is bounded

Thm 3.14: If a non empty set K is compact, then is closed and bounded.

Proof: (i) Suppose K is compact.

Let's prove K is bounded.



Fix some $x \in K$

$\{B^\circ(x,n)\}_{n\in\mathbb{N}}$ is a cover for K

Since $\bigcup_{n=1}^{\infty} B^\circ(x,n) = X \supset K$

Since K is compact this cover has a finite subcover:

$B^\circ(x,n_1), B^\circ(x,n_2), \dots, B^\circ(x,n_m)$

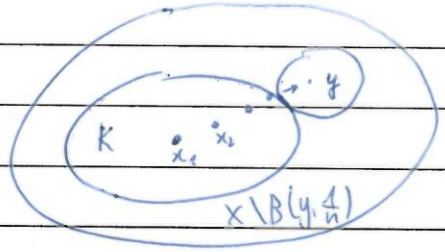
$N = \max\{n_1, \dots, n_m\}$

$\Rightarrow K \subset B^\circ(x,N)$

$\forall\, x,y \in K$, $d(y,z) \le d(y,x) + d(x,z) \le 2N$

$\operatorname{diam}(K) \le 2N \Rightarrow K$ is bounded.

(2) Suppose K is compact.
Let's prove it's closed
Suppose K is not closed
This means $\exists\, y \in X \setminus K$
and a sequence $x_n \in K \to y$



Consider $\{\underbrace{X \setminus B(y, \tfrac{1}{n})}_{\text{open}}\}_{n \in \mathbb{N}}$

$\bigcup_{n=1}^{\infty} \left(X \setminus B(y, \tfrac{1}{n}) \right) = X \setminus \{y\} \supset K \Rightarrow$ So this is a cover for K.

Since K is compact this cover has a finite subcover
$X \setminus B(y, \frac{1}{n_1}), \dots \; X \setminus B(y, \frac{1}{n_m})$
$N = \max\{n_1, \dots n_m\} \Rightarrow K \subset X \setminus B(y, \frac{1}{N})$
$\Rightarrow$ all $x_n \in X \setminus B(y, \frac{1}{N}) \Rightarrow d(x_n, y) > \frac{1}{N}$
$\Rightarrow d(x_n, y) \not\to 0$ $\square$
contradiction.

Is the converse statement true?
closed + bounded $\Rightarrow$ compact.
yes in $\mathbb{R}^n$ with any norm
NO discrete space, infinite set
↳ not compact.
↳ closed
↳ bounded.

19-3-20..

// A set K is compact if any cover $\{G_\alpha\}_{\alpha \in A}$ of K has
open set.

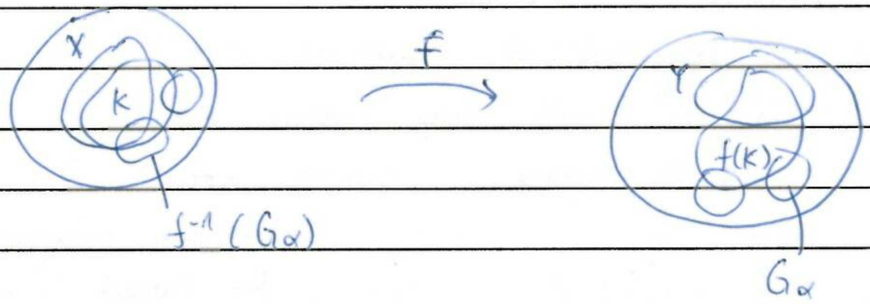
Compact: $[a,b]$ (Heine-Borel)
finite set in a discrete space

Theorem: compact $\Rightarrow$ closed and bounded
compact $\overset{?}{\Leftarrow}$ closed and bounded.

Yes: in $\mathbb{R}^n$ with Euclidean norm - without proof

No: In general (for example: K = infinite set in a discrete space)

// Thm 3.15: Let (X, d_x) and (Y, d_y) be two metric spaces.
Let $f: X \to Y$ be a continuous function.
Let $K \subset X$ be a compact set.



Then $f(K)$ is a compact set in Y.

Proof: Let $\{G_\alpha\}_{\alpha \in A}$ be a cover of $f(k)$
Consider $\{f^{-1}(G_\alpha)\}_{\alpha \in A}$
Since each G_α is open $\Rightarrow$ each $f^{-1}(G_\alpha)$ is open
so $\{f^{-1}(G_\alpha)\}_{\alpha \in A}$ is a cover of K
Since K is compact there is a finite subcover
$f^{-1}(G_{\alpha_1}), \dots f^{-1}(G_{\alpha_m})$
$\Rightarrow$ $G_{\alpha_1}, \dots G_{\alpha_m}$ is a cover of $f(k)$
So this is a finite subcover $\Rightarrow$ $f(k)$ is compact. $\square$

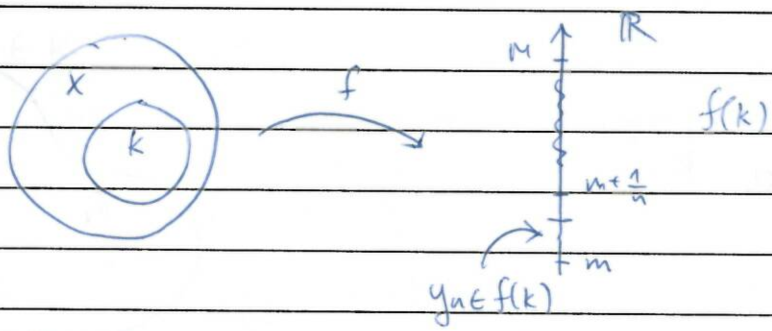
// Thm 3.16 : Let (X, d_X) be a metric space
$f : X \to \mathbb{R}$ be a continuous function.
$K \subset X$ be a compact set.

Then f is bounded on K
and if $m = \inf_{x \in K} f(x)$

$M = \sup_{x \in K} f(x)$

Then there are $x_1, x_2 \in K$ st. $f(x_1) = m$, $f(x_2) = M$.

Proof :



Since K is compact and f is continuous $\Rightarrow$ $f(k)$ is bounded.

$\Rightarrow$ $-\infty < m, M < \infty$ (both finite)

$\forall n$ $m + \frac{1}{n}$ is not a cover bound of $f(k)$ $\Rightarrow$

$\exists\, y_n \in f(k)$ st. $m \le y_n \le m + \frac{1}{n}$

Since K is compact and f is continuous $\Rightarrow$

$f(k)$ is closed, So all $y_n \in f(k) \Rightarrow m \in f(k)$

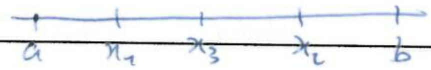
$\Rightarrow$ $\exists x_1 \in K$ st. $f(x_1) = m$

The proof of M is similar

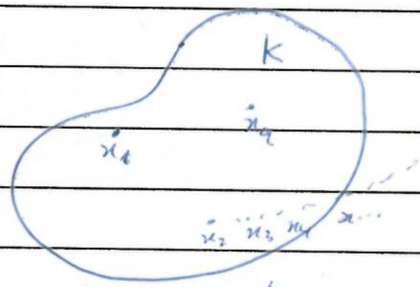
// Def^n : let (X,d) be a metric space
A set $K \subset X$ is sequentially compact
if any sequence of points in K has a subsequence which converges to a point in K.

Examples :

(1) ~~[illegible]~~ $(\mathbb{R}, |\cdot|)$; $K = [a,b]$
Bolzano - Weierstrass
$[a,b]$ is sequentially compact.

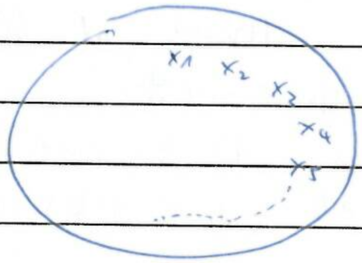


(2) Discrete space
(a) finite set K



sequentially compact.

(b) infinite set K.



Take all x_n different. Any sequence keeps this property
$\Rightarrow$ doesn't converge (only eventually consist. sequences converge in discrete space) .

// Thm 3.17 : K is compact $\Leftrightarrow$ K is sequentially compact.

without proof

proof.

Proof : ($\Rightarrow$)

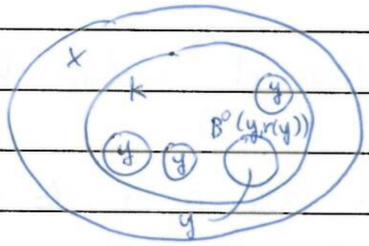
Suppose K is compact.

But suppose K is not sequentially compact.

$\Rightarrow$ $\exists (x_n)$ in K which has no subsequence converging to a point in K.

Fixed $y \in K$

(x_n) has no subsequence converging to y.



$\Rightarrow \exists r(y)$ st. $B^\circ(y, r(y)) \setminus \{y\}$

Contains no points of the sequence (x_n)

$\{B^\circ(y, r(y))\}_{y \in K}$ - cover of K

$B^\circ(y_1, r(y_1)), \dots B^\circ(y_m, r(y_m))$

(x_n) takes values in the set.

$\{y_1, \dots y_m\}$

$\Rightarrow$ one of the values y_i is take infinitely many times

$\Rightarrow$ this gives a convergent sequence to $y_i \in K$

$\Rightarrow$ contradiction.

$\square$

22-3

// K- compact $\Leftrightarrow$ every cover has a finite subcover

K- sequentially compact $\Leftrightarrow$ any sequence $\{x_n\}$ in K has subsequence converging to a point in K.

[compact = sequentially compact.]

compact $\Rightarrow$ ~~false~~ closed + bounded.

compact $\Leftarrow$ closed + bounded.

yes : in $\mathbb{R}^n$ with Euclidean norm

no : in general (for example in a discrete space)

What is the answer if we are in a normed space?

** No : $C[0,1]$, $\|\cdot\|_{sup}$

$B(0,1)$ - closed ball of radius 1
$\uparrow$ zero function.

- closed (as every closed ball is a closed set)
- bounded : $f, g \in B(0,1)$

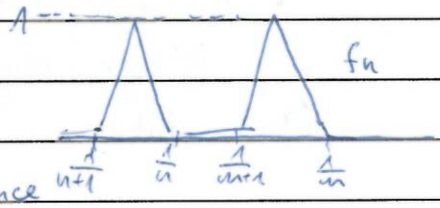
$$\|f-g\|_{sup} \leq \|f\|_{sup} + \|g\|_{sup} \leq 1+1=2$$

$\operatorname{diam} B(0,1) \leq 2$

- not sequentially compact :

$\{f_n\}$

$\|f_n - f_m\|_{sup} = 1$ if $m \neq n$



$\Rightarrow$ $\{f_n\}$ has no convergence subsequence

- not compact since not sequentially compact.

// Def^n: Let V be a vector space and $\|\cdot\|$ & $|\cdot|$ be two norms on V. The norms $\|\cdot\|$ and $|\cdot|$ are equivalent if there are $c, C > 0$ st.

$$c|x| \leq \|x\| \leq C|x|, \quad \forall x$$

Examples: ① $\mathbb{R}^n$, $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\max_{1 \leq i \leq n} |x_i| \leq \underbrace{\sum_{i=1}^{n} |x_i|} \leq n \cdot \max_{1 \leq i \leq n} |x_i|$$

$$1 \cdot \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$$

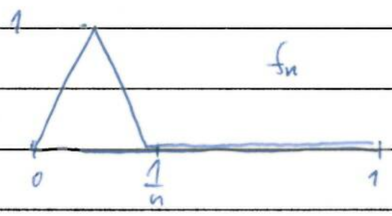
$\Rightarrow$ these norms are equivalent.

②: $C[0,1]$ $\quad \|f\|_{sup} = \sup_{x \in [0,1]} |f(x)|$, $\quad \|f\|_1 = \int_0^1 |f(x)| \, dx$ $\Big\}$ not equivalent!

Suppose they are equivalent:

$$c\|f\|_1 \leq \|f\|_{sup} \leq C\|f\|_1 \quad \forall f$$

$$\Rightarrow \underbrace{c\|f_n\|_1}_{\frac{1}{2n} \to 0} \leq \underbrace{\|f_n\|_{sup}}_{1} \leq \underbrace{C\|f_n\|_1}_{\frac{1}{2n} \to 0}$$



$\therefore$ contradiction by Sandwich Thm.

// ~~Claim~~ Claim $\|\cdot\|$ and $|\cdot|$ are equivalent

$\Updownarrow$

They have the same set of convergent sequences.
In particular, they have the same closed sets, ...
op...
c...

Claim: Any two norms on $\mathbb{R}^n$ are equivalent!
In particular, any norm is equivalent to t...

// Def^n: Let V be a vector space and $\|\cdot\|$ & $|\cdot|$ be two norms on V. The norms $\|\cdot\|$ and $|\cdot|$ are equivalent if there are $c, C > 0$ st.

$$c|x| \leq \|x\| \leq C|x| \, , \forall x$$

Examples: ① $\mathbb{R}^n$, $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\underbrace{\max_{1 \leq i \leq n} |x_i| \leq \sum_{i=1}^{n} |x_i|}_{} \leq n \cdot \max_{1 \leq i \leq n} |x_i|$$

$$1 \cdot \|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$$

$\Rightarrow$ these norms are equivalent.

②: $C[0,1]$ $\quad \|f\|_{sup} = \sup_{x \in [0,1]} |f(x)|$, $\quad \|f\|_1 = \int_0^1 |f(x)|\, dx$ $\Big\}$ not equivalent!

Suppose they are equivalent:

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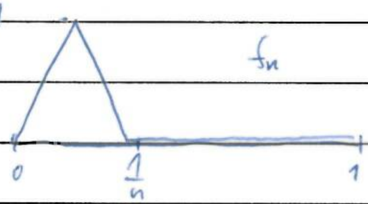
$$\Rightarrow \underbrace{c\|f_n\|_1}_{\frac{1}{2n} \to 0} \leq \underbrace{\|f_n\|_{sup}}_{1} \leq \underbrace{C\|f_n\|_1}_{\frac{1}{2n} \to 0}$$



$\therefore$ contradiction by Sandwich Thm.

// ~~Claim~~ Claim $\|\cdot\|$ and $|\cdot|$ are equivalent

$\Updownarrow$

They have the same set of convergent sequences. In particular, they have the same closed sets, ...
open sets, ...
compact sets ...

Claim: Any two norms on $\mathbb{R}^n$ are equivalent!
In particular, any norm is equivalent to the Euclidean norm.

EXAM

- 2 qus in Ch 1
- 1 - Ch 2
- 3 - Ch 3